APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2024 (May)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: May 22, 2024

Time: 9:00 AM – 1:00 PM

A1. Consider the conservation law

$$u_t + (u^2)_x = 0,$$

with the initial condition

$$u(x,0) = \begin{cases} 0 & x \le 0 \\ 2 & 0 < x \le 4 \\ 1 & x > 4 \end{cases}.$$

(a). Find and draw the characteristics of the equation.

(b). Find the solution at $t = 1,\ 2$.

(c). Solve the Riemann problem with initial condition

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}.$$

(d). If the initial condition is given as

$$u(x,0) = \frac{1}{1+x^2},$$

find the breaking time at which a shock starts to form.

Continue solution:

A2. Using the separation of variables method, solve the following boundary value problem

$$u_{tt} - 4u_{xx} = 0, \quad x \in (0,1)$$

$$u(0,t) = 1, \quad u(1,t) = 2, \quad u(x,0) = 1 + 2x - x^2, \;\; u_t(x,0) = 0.$$

Continue solution:

A3. Find the Green's function of second kind in two dimensional domain

$$\Delta G(x,y) = \delta(x-y), \quad x,y \in \Omega,$$

$$\left.\frac{\partial G}{\partial n}\right|_{\partial\Omega} = 0, \quad \Omega : \{x = (x_1, x_2)\} \in \mathcal{R}^2,\ x_1 > x_2 > -x_1,\ x_1 > 0\}$$

Continue solution:

A4.

(a) Given the real integral

$$I = \int_0^\infty \frac{x^{a-1}}{x^2+1} dx, \quad a \in \mathbb{R},$$

determine the range of the parameter a that ensures the existence of I.

(b) Compute I in terms of a.

Continue solution:

A5.

(a) Find a linear fractional transformation that maps the points $z_1 = -1$, $z_2 = 0$, and $z_3 = i$ in the z-plane into the respective points $w_1 = 0$, $w_2 = 1$, ans $w_3 = \infty$ in the w-plane.

(b) If $\mathcal{D}$ is the interior of a circle that contains points z_1, z_2, and z_3, find the image of $\mathcal{D}$ in the w-plane under this transformation.

Continue solution:

B6. Consider the problem:

$$\min_{\boldsymbol{x}} f(\boldsymbol{x}) = 3x_1^2 - 4x_1x_2 + x_2^2 - 2x_3^2 + 5x_3$$

subject to

$$g(\boldsymbol{x}) = x_1 + 3x_2 - x_3 - 3 = 0.$$

a) **(2 points)** Compute the Lagrange function $\mathcal{L}(\boldsymbol{x}, \lambda)$ associated with this constrained minimization problem.

b) **(3 points)** Derive the system of equations necessary to find the critical points of the Lagrange function. You do not need to solve the system.

c) **(5 points)** Determining whether the critical point $\boldsymbol{x}^*$ is a constrained local minimum. You do not need to determine $\boldsymbol{x}^*$ first.

Continue solution:

B7. Consider the initial value problem given by:

$$y' = -\alpha t y^3 + \beta \sin(ty), \quad y(0) = 1,$$

where the independent variable t ranges from 0 to 1 and α and β are positive constants.

(a) **(3 points)** Apply an explicit second-order numerical method of your choice for solving the given initial value problem. Provide details on the method you choose and the rationale behind this choice.

(b) **(4 points)** Apply an implicit second-order numerical method suitable for handling the potentially stiff nature of the given differential equation when α and β are large. Provide details of your chosen method along with its representation. Discuss briefly why your chosen method might be preferable for stiff or challenging differential equations.

(c) **(3 points)** Discuss the challenges associated with solving the nonlinear system that arises from the implicit method developed in part (b). Explain how the nonlinear system can be solved, identify potential challenges, and propose strategies to address these challenges.

Continue solution:

B8. Consider a nonlinear hyperbolic scalar conservation law $v_t + [f(v)]_x = 0$.

(a) Integrate this PDE in space over one grid cell $[x_{k-1/2}, x_{k+1/2}]$ and in time over the interval $[t^n, t^{n+1}]$ and derive components of the conservative numerical scheme

$$u_k^{n+1} = u_k^n - \frac{\Delta t}{\Delta x}[F(u_k^n, u_{k+1}^n) - F(u_{k-1}^n, u_k^n)].$$

(b) Derive numerical fluxes of the Godunov method.

(c) Formulate requirements for a numerical method in conservative form to be consistent with the corresponding PDE.

(d) Explain how using a conservative numerical scheme leads to global conservation of the quantity u.

Continue solution:

B9. Derive a second-order numerical scheme for the linear advection equation $v_t + av_x = 0$ using the following approach:

(a) Assume that the cell averages u_k^n at time step n are used to perform a piece-wise linear reconstruction $\tilde{u}^n(x) = u_k^n + \sigma_k(x - x_k)$ in each cell $x \in [x_{k-1/2}, x_{k+1/2}]$, where the slopes σ_k are computed as $\sigma_k^n = \frac{u_{k+1}^n - u_k^n}{\Delta x}$.

(b) Find exact solution to the linear advection equation at time t^{n+1} using the above piece-wise linear function as the initial condition.

(c) Compute cell averages at time t^{n+1} and show that the resulting numerical scheme is equivalent to the Lax-Wendroff scheme (you may consider a specific sign of a, e.g. $a > 0$, to simplify calculations).

(d) Based on the selection of slopes in the piece-wise linear reconstruction step (illustrate graphically), explain why the Lax-Wendroff scheme is not TVD.

(e) Propose a slope-limiter for the reconstruction step to achieve TVD properties.

Continue solution:

B10. Using the discrete Fourier transform, perform stability analysis of the FTCS scheme for the two-dimensional inhomogeneous diffusion equation $v_t = \nu(v_{xx} + v_{yy}) + bv$, where ν and b are positive constants.

Continue solution: