Applied Mathematics and Statistics

Foundation Qualifying Examination Part B

in Computational Applied Mathematics

Spring 2020 (January)

(Closed Book Exam)

Please solve 3 out of 4 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part B: 1 2 3 4

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 27, 2020

Time: 11:15 AM – 13:15 PM

B1. (10 points) Consider the following differential equation

$$(x-1)y'' - xy' + y = 0.$$

a) Find and classify all singular points.

b) Obtain two linearly independent solutions at x = 1 using the method of Frobenius.

c) Use your results to prove that a Taylor serier expansion of any solution to this differential equation about $x = 0$ has an infinite radius of convergence.

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B2. (10 points) Consider the following initial value problem (IVP) with a small positive parameter ε

$$y'' + (1+\varepsilon)y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

a) Obtain a first order perturbative approximation $y(x) = y_0(x) + \varepsilon y_1(x)$ to this IVP.

b) Find the exact solution to the IVP.

c) Compare the behavior of the perturbative solution at large x with the exact solution. In which x-domain is this approximate solution valid?

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B3.

Let $A \in \mathbb{R}^{m \times n}$ be a matrix with rank $n-1$, where $m \geq n$. Suppose $AP = QR$ is the reduced QR factorization of A with column pivoting, where P is a permutation matrix, such that the diagonal entries of R are non-increasing in magnitude, i.e., $|r_{11}| \geq |r_{22}| \geq \cdots \geq |r_{nn}|$. Ignore the effect of rounding errors.

a) (5 points) Show that $r_{nn} = 0$.

b) (5 points) Express the null space of A^T in terms of Q.

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B4.

Let $A = \begin{bmatrix} I & B \\ B^T & I \end{bmatrix}$, where $B \in \mathbb{R}^{m\times m}$ with $\|B\|_2 < 1$.

a) (5 points) Show that the columns of the matrix

$$X = \begin{bmatrix} U & U \\ V & -V \end{bmatrix}$$

are the eigenvectors of A, where $B = U\Sigma V^T$ is a singular value decomposition of B. What are the corresponding eigenvalues?

b) (5 points) Show that A is symmetric and positive definite, and its condition number in 2-norm is

$$\kappa(A) = \frac{1 + \|B\|_2}{1 - \|B\|_2}.$$

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