

Applied Mathematics and Statistics
Foundation Qualifying Examination Part B
in Computational Applied Mathematics
Spring 2022 (January)

Date of Exam: January 20, 2022, **Time:** 11:15 AM – 13:15 PM

Instructions: There are 3 problems, and you are required to solve all of them. All problems are weighted equally. Please show detailed work for full credit.

- This exam is conducted via Zoom on January 20 from 11:15 am to 1:15 pm EST.
- The entire Zoom meeting and chat messages are being recorded.
- This is a closed book, closed note exam.
- Hand calculators (or other computing devices) may not be used during the exam.
- You should join the Zoom meeting from two devices: Your computer/laptop/tablet (with webcam), and your smartphone (with camera).
- Audio should be muted and video must be kept on during the exam.
- Your computer webcam must fully show your face; your smartphone camera should show your computer monitor, your hands and workspace, with the pages of paper being used for the exam.
- At the very beginning of the exam, during set up, you will be asked to do a brief “environment scan”, showing the workspace where your computer is and the desk/table/floor where you will be writing your work.
- You are required to bring enough blank pieces of paper to use for the exam. You will show the blank pages at the beginning, during the “environment scan” on Zoom.
- You are not allowed to use the internet for any searching or communication with others, with the sole exception of communication privately with the proctors via Zoom chat (which is set so that your chats only go privately to hosts, not to others).
- After you finish the exam, scan your pages, numbered, ordered and oriented appropriately, into a single pdf file. Email the pdf file to **roman.samulyak@stonybrook.edu** no later than 5 minutes after completion of the exam (i.e., **by 1:20 pm EST**).
- No students are allowed to leave the Zoom meeting until the exam is over.
 - If you finish the exam early, then submit your exam and remain in the Zoom meeting until the conclusion of the exam at 1:15 pm EST.
 - After submitting your exam, you can study for another exam or work on anything else, while staying in view in the Zoom meeting.
- If the answers are not submitted by 1:20 pm EST, the exam will not be graded, and a score of zero will be given.
- If you have a question during the exam, then send a chat message to the host privately.

B1. Exploring the relation between the Riccati equation and a second order linear differential equation, describe the behavior of solutions at large positive x -values for the following Riccati equation

$$y' = y^2 + x.$$

In particular

- Convert the Riccati equation to a second order linear equation using $y(x) = -v'(x)/v(x)$.
- Describe singular points and find the controlling factor of the leading behavior of solutions to the second order equation as $x \rightarrow +\infty$. (Finding the controlling factor is sufficient - do not attempt to improve it.)
- Using this information, obtain (real-valued) solutions to the Riccati equation and illustrate them graphically.
- Compute the distance between consecutive singularities of solutions at large positive x -values.

B2. Consider the following boundary layer problem

$$\varepsilon y'' - \frac{1}{x} y' - y = 0, \quad 0 < x < 1, \quad y(0) = 1, \quad y(1) = 1.$$

- Discuss the location and thickness of the boundary layer.
- Introduce the inner variable and obtain the inner and outer solutions in the 0-order approximation.
- Perform asymptotic matching and find the uniform solution (express it in terms of the original variable x).

B3. Consider the matrix $C = (A^T A)^{-1}$, where $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = n$.

- (3 points) Suppose the reduced QR factorization $A = QR$ is available. Show that $C = (R^T R)^{-1}$.
- (4 points) Show that the spectral radius of A^{-1} is no greater than the square root of the 2-norm of C , i.e., $\rho(A^{-1}) \leq \sqrt{\|C\|_2}$.
- (3 points) What is the condition number of C in 2-norm in terms of the singular values of A ?