Holographic Mellin Amplitudes

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Xinan Zhou

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Xinan Zhou

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Leonardo Rastelli - Dissertation Advisor
Professor, Department of Physics and Astronomy

Peter van Nieuwenhuizen - Chairperson of Defense
Distinguished Professor, Department of Physics and Astronomy

George Sterman
Distinguished Professor, Department of Physics and Astronomy

Martin Roček
Professor, Department of Physics and Astronomy

Thomas Allison
Assistant Professor, Department of Physics and Astronomy, and Chemistry

Michael Anderson
Professor, Mathematics Department

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

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This dissertation focuses on developing efficient modern methods to compute holographic four-point correlators in superconformal field theories in various spacetime dimensions.

Three approaches are presented in this dissertation, which are inspired by the bootstrap philosophy and the on-shell methods of scattering amplitudes in flat space. I first review the inherent difficulties of the traditional method, and suggest an improved algorithm that drastically simplifies the calculation. I further show that by translating the problem into Mellin space many difficulties encountered in position space are avoided. Holographic correlators become Mellin amplitudes in Mellin space, which are the natural analogue of the flat space S-matrix. I argue that imposing constraints from superconformal symmetry and general consistency conditions is enough to fix the Mellin amplitude, avoiding all details of the complicated effective Lagrangian. I develop two complementary Mellin space techniques, and obtain many novel results for holographic four-point functions in $AdS_5 \times S^5$, $AdS_7 \times S^4$ and $AdS_4 \times S^7$.

To my parents

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List of Publications

[1] **"Mellin Amplitudes for Supergravity on $\mathbf{AdS_5 \times S^5}$"**,
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[6] **"On Superconformal Four-Point Mellin Amplitudes in Dimension $\mathbf{d > 2}$"**,
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[7] **"On Mellin Amplitudes in SCFTs with Eight Supercharges"**,
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[1] The content of this publication is not discussed in this thesis.

Chapter 1

Introduction

Arguably the development of Quantum Field Theory (QFT) – a mathematical framework that unifies Quantum Mechanics and Special Relativity – is one of the biggest triumphs of the 20^{th} century physics. This versatile framework underlies many branches of modern physics research. For example, it is the cornerstone of the most successful theory to date, the Standard Model, whose accuracy has been tested down to a scale at least $1/1000^{th}$ the size of an atomic nucleus. In Condensed Matter Physics, QFT provides effective description for systems at long distances. QFT is also useful in cosmology, for example to describe early universe inflation. However our understanding of QFT is still largely incomplete: most of the knowledge is limited in the realm where the constituents of the system are interacting weakly (the weak coupling limit) so that we can treat the system perturbatively around its (trivial) free limit. At strong coupling, on the other hand, we have no systematic method to make progress. Taming QFTs at strong coupling is an urgent task for 21^{th} century physicists, and its importance cannot be overemphasized by a long list of important unsolved problems in modern-day physics. We just name a few here: the generation of a mass gap in Yang-Mills theory (one of the seven Millennium Prize Problems defined by the Clay Mathematics Institute); the confinement of quarks and chiral symmetry breaking in Quantum Chromodynamics; and the mechanism of high temperature superconductors.

Recent years have witnessed some promising progress towards understanding strongly coupled QFTs. One is the revival of the old bootstrap philosophy [8] stemmed from the work of Rattazzi, Rychkov, Tonni and Vichi [9]. The idea of the bootstrap program is to extract the full physical content of a theory – even at strong coupling – solely based on symmetries, self-consistency conditions and a small amount of additional physical input, without resorting to any approximation. This line of attack is most effective on theories constrained by a large amount of symmetries. Among them are Conformal Field Theories (CFTs), which are QFTs with additional conformal symmetry and describe the critical behavior of second order phase transitions. The most spectacular highlight of bootstrap's modern comeback is the numerical solution of the three-dimensional Ising model at its critical point [10, 11], whose precision eclipses any other existing methods. Another breakthrough comes from a remarkable conjecture that goes under the name of AdS/CFT duality or holography principle [12, 13, 14]. This conjecture equates strongly coupled conformal field theories to gravitational theories in a higher dimensional spacetime, and has passed numerous non-trivial checks. Taking the duality conjecture as a truth, we can obtain analytic results in strongly coupled theories by doing only perturbative calculations in the dual side.

The goal of this dissertation is to combine the bootstrap idea with holography to gain analytic insight into strongly coupled QFTs. We aim to compute analytically correlation functions – the physical observables that quantify how fluctuations at different spacetime points are correlated – in various strongly interacting theories in different spacetime dimensions. By analyzing these correlators, a wealth of information can be extracted. Holographically, the correlation functions of the boundary theory are represented by scattering amplitudes in a curved bulk spacetime, the Anti-de-Sitter space, sourced by boundary fluctuations. Computing these AdS space scattering amplitudes has a long history [15, 16] that can be tracked back to the beginning of the AdS/CFT correspondence. There is a straightforward algorithm based on perturbative expansion of the effective Lagrangian and correlators are computed as sums of Feynman diagrams in AdS space (the Witten diagrams). However due to the extraordinary complexity, this algorithm quickly turned out to be inadequate. Even for the most canonical duality pair, namely four dimensional $\mathcal{N}=4$ Super Yang-Mills theory at infinite 't Hooft coupling and IIB supergravity on $AdS_5 \times S^5$, the computation of all the *four*-point functions[1] remained a longstanding problem for twenty years[2]. In this dis-

[1]Two-point functions and three-point functions are trivial as their structures are completely fixed by symmetry.

[2]Prior to publication [1].

sertation, we present modern methods which circumvent the difficulties of the traditional method. These methods allow us to efficiently compute holographic correlators and have led to a number of novel results. The most noticeable outcome of our endeavor is an elegantly simple solution of *all* one-half BPS four-point functions for the paradigmatic $\mathcal{N}=4$ Super Yang-Mills theory. The methods described here are inspired by the bootstrap program: we compute the holographic correlators by using only symmetry principles and consistency conditions, avoiding all details of the complicated effective Lagrangian. Our results also share a lot of similarities with the on-shell scattering amplitude in flat space. This analogy becomes particularly clear after we use the Mellin representation formalism, in which the correlators become Mellin amplitudes. The similar intricate structures in Mellin amplitudes makes it tantalizing to contemplate the possibility of having a full-fledged program of AdS scattering amplitudes that parallels the highly successful paradigm in flat space. The progress reported in this dissertation can be viewed as a modest first step towards such a systematic understanding.

In the following, we briefly review the theories in this dissertation for which we will compute correlators, and summarize our main results.

Four-Dimensional $\mathcal{N}=4$ Super Yang-Mills Theory

$SU(N)$ $\mathcal{N}=4$ Super Yang-Mills theory can be viewed as the supersymmetric generalization of the $SU(3)$ Yang-Mills theory that governs the strong interaction, with the maximal amount of supersymmetry allowed in four dimensions. The theory is further conformal, which enhances the global symmetry to be the superconformal group $PSU(2,2|4)$. Holographically, $\mathcal{N}=4$ SYM is dual to type IIB string theory on $AdS_5\times S^5$ background. In the limit of taking N large and further sending the 't Hooft coupling $\lambda=g^2_{YM}N$ to infinity, the AdS theory reduces to IIB supergravity.

In this dissertation, we are interested in computing the four-point functions of the one-half BPS operators (see Section 2.1 for their definition). The holographic computation for these four-point functions has only been performed for a handful of cases due to unsurmountable computational complexities: three cases of equal weight correlators [17, 18, 19] and correlators with the "next-next-to-extremal" configuration [20, 21, 22]. A conjecture for correlators with arbitrary equal weights k was made in [23]. In Section 2.3 we introduce a "position space method" which overcomes some of the difficulties of the traditional algorithm. Using this method, we reproduced the

$k = 2, 3, 4$ results in [17, 18, 19] and computed a new $k = 5$ case which is in agreement with the conjectural expression in [23]. A more elegant solution comes from formulating an algebraic bootstrap problem in Mellin space (Chapter 3), by combining symmetry constraints and self-consistency conditions (Section 4.1). The solution to this algebraic problem gives all the one-half BPS four-point functions (Section 4.2).

Six-Dimensional $\mathcal{N} = (2, 0)$ SCFTs

The existence of non-trivial fixed point in dimension higher than four is quite remarkable from the Renormalization Group point of view. The most famous, yet mysterious, conformal field theories in six dimensions are the $\mathcal{N} = (2, 0)$ superconformal CFTs. These strongly interacting theories can be realized as the low energy effective description of n coinciding M5-branes, and have the maximal amount of superconformal symmetry $OSp(8^*|4)$ in six dimensions.[3] At large n, the SCFT can be equivalently described by a Kaluza-Klein supergravity theory on $AdS_7 \times S^4$ that comes from the reduction of eleven dimensional supergravity. The only four-point function computed for this background is the four-point function of the stress-tensor multiplet [25]. We reproduced this result and further computed massive KK correlators using the position space method. We also set up an algebraic bootstrap in Mellin space (Section 4.3) and gave some of the solutions to this problem (Section 4.4). Moreover, we solved all the next-next-to-extremal correlators in Mellin space using another technique from Chapter 5.

Three-Dimensional Aharony-Bergman-Jafferis-Maldalcena Theories

The Aharony-Bergman-Jafferis-Maldalcena (ABJM) Theories [26] are Chern-Simons-matter theories in three dimensions with gauge group $U(N)_k \times U(N)_{-k}$ and $k = 1, 2$. These theories have $\mathcal{N} = 8$ superconformal symmetries and describe the effective theory on N coinciding M2-branes. The $k = 1$ theories were conjectured to be dual to eleven dimensional supergravity on $AdS_4 \times S^7$, in the limit of large N. Though of tremendous physical interest, no four-point functions had ever been computed for this background. This is partly due to

[3] By a classic result of Nahm [24], this is also the largest spacetime dimension that allows superconformal field theories.

the technical difficulty that exchange diagrams in the traditional algorithm cannot be evaluated in closed form (see Section 2.1), which renders the position space method from Section 2.3 ineffective as well. What's worse, this theory is also inaccessible by the Mellin space method in Chapter 4. To this end, we introduced a complementary approach in Chapter 5, which allowed us to compute the first four-point function in this theory (Section 5.2).

Outline of the Thesis

The rest of this dissertation is organized as follows. In Chapter 2 we discuss the computation of correlators in position space. After reviewing the difficulties of the traditional algorithm in Section 2.1, we introduce an improved position space method in Section 2.3. This method is applied in Section 2.3.1 to compute correlators in $AdS_5 \times S^5$ and $AdS_7 \times S^4$. In Chapter 3 we introduce the Mellin representation formalism. In Section 3.1 we review the formalism for a general CFT. In Section 3.2 we make a small digression to extend this formalism to include CFTs with a conformal boundary or a defect (interface). In Chapter 4 we introduce a Mellin space method which translates the task of computing four-point functions into solving an algebraic bootstrap problem. The problem is set up for $AdS_5 \times S^5$ in Section 4.1 and the full solution is given in Section 4.2. A similar problem is set up for $AdS_7 \times S^4$ in Section 4.3 and partial solutions are given in Section 4.4. In Chapter 5 we introduce another Mellin space technique that is complementary to the method in Chapter 4. We discuss the application of this method to various backgrounds in Section 5.2. Technical details of this dissertation have been relegated to the four appendices.

The content of dissertation has appeared in the papers [1, 3, 4, 5, 6].

Chapter 2

Computations in Position Space

2.1 The Traditional Algorithm

The standard recipe to calculate holographic correlation functions follows from the most basic entry of the AdS/CFT dictionary [13, 12, 14], which states that the generating functional of boundary CFT correlators equals the AdS path integral with boundary sources. Schematically,

$$\left\langle e^{i\int_{\partial AdS}\bar{\varphi}_i\mathcal{O}_i}\right\rangle_{\mathrm{CFT}} = Z[\bar{\varphi}_i] = \int_{AdS}\mathcal{D}\varphi_i\; e^{iS}\,\Big|_{\varphi_i\big|_{z\to 0}=\bar{\varphi}_\Delta}\,. \tag{2.1}$$

Here and throughout the thesis we use the Poincaré coordinates

$$ds^2 = R^2\,\frac{dz^2 + d\vec{x}^2}{z^2}\,. \tag{2.2}$$

The AdS radius R will be set to one by a choice of units, unless otherwise stated.

We focus on the limit of the duality where the bulk theory becomes a weakly coupled gravity theory. To make the following discussion concrete, let us take the canonical duality pair of $\mathcal{N}=4$ SYM and type IIB string theory on $AdS_5\times S^5$. This limit of the duality amounts to taking the number of colors N large and further sending the 't Hooft coupling $\lambda = g_{YM}^2 N$ to infinity. In this limit, the bulk theory reduces to IIB supergravity with a small five-dimensional Newton constant $\kappa_5^2 = 4\pi^2/N^2 \ll 1$. The task of computing correlation functions in the strongly coupled planar gauge theory has thus become the task of computing suitably defined "scattering amplitudes" in the weakly coupled supergravity on an AdS_5 background. The AdS supergravity amplitudes can be computed by a perturbative diagrammatic expansion, in powers of the small Newton constant, where the so-called "Witten diagrams" play the role of position space Feynman diagrams. The Witten diagrams are "LSZ reduced", in the sense that their external legs (the bulk-to-boundary propagators) have been put "on-shell" with Dirichlet-like boundary conditions at the boundary ∂AdS_{d+1}.

We restrict ourselves to the evaluation of four-point correlation functions of the single-trace one-half BPS operators,

$$\mathcal{O}^{(k)}_{I_1\ldots I_k} \equiv \mathrm{Tr} X^{\{I_1} \ldots X^{I_k\}}\,, \quad k \geqslant 2\,, \tag{2.3}$$

where X^I, $I = 1, \ldots 6$ are the scalar fields of the SYM theory, in the $\mathbf{6}$ representation of $SO(6) \cong SU(4)$ R-symmetry. The symbol $\{\ldots\}$ indicates the projection onto the symmetric traceless representation of $SO(6)$ – in terms of $SU(4)$ Dynkin labels, this is the irrep denoted by $[0, k, 0]$. In the notations of [27], the operators (2.3) are the superconformal primaries of the one-half BPS superconformal multiplets $\mathcal{B}^{(\frac{1}{2},\frac{1}{2})}_{[0,k,0]}$. They are annihilated by half of the Poincaré supercharges and have protected dimensions $\Delta = k$. By acting with the other half of the supercharges, one generates the full supermultiplet, which comprises a finite number of conformal primary operators in various $SU(4)$ representations and spin $\leqslant 2$ (see, *e.g.*, [27] for a complete tabulation of the multiplet). Each conformal primary in the $\mathcal{B}^{(\frac{1}{2},\frac{1}{2})}_{[0,k,0]}$ multiplet is dual to a supergravity field in AdS_5, arising from the Kaluza-Klein reduction of IIB supergravity on S^5 [28], with the integer k corresponding to the KK level. For example, the superprimary $\mathcal{O}^{(k)}$ is mapped to a bulk scalar field s_k, which is a certain linear combination of KK modes of the $10d$ metric and four-form with indices on the S^5.

The traditional method evaluates the correlator of four operators (2.3) as the sum of all tree level diagrams with external legs s_{k_1}, s_{k_2}, s_{k_3}, s_{k_4}. One needs the precise values of the cubic vertices responsible for exchange diagrams (Figure 2.1), and of the quartic vertices responsible for the contact diagrams (Figure 2.2). The relevant vertices have been systematically worked out in the literature [29, 30, 31, 32] and take very complicated expressions. Our methods in this thesis, on the other hand, do not require the detailed form of these vertices, so we will only review some pertinent qualitative features.

Let us first focus on the cubic vertices. The only information that we need are selection rules, *i.e.*, which cubic vertices are non-vanishing. An obvious constraint comes from the following product rule of $SU(4)$ representations,

$$[0, k_1, 0] \otimes [0, k_2, 0] = \sum_{r=0}^{\min\{k_1,k_2\}} \sum_{s=0}^{\min\{k_1,k_2\}-r} [r, |k_2 - k_1| + 2s, r] \,, \tag{2.4}$$

which restricts the $SU(4)$ representations that can show up in an exchange diagram. We collect in Table 2.1 (reproduced from [27, 19]) the list of bulk fields $\{\varphi_{\mu_1 \ldots \mu_\ell}\}$ that are *a priori* allowed in an exchange diagram with external s_{p_i} legs if one only imposes the R-symmetry selection rule.

fields	s_k	$A_{\mu,k}$	$C_{\mu,k}$	ϕ_k	t_k	$\varphi_{\mu\nu,k}$
irrep	$[0,k,0]$	$[1,k-2,1]$	$[1,k-4,1]$	$[2,k-4,2]$	$[0,k-4,0]$	$[0,k-2,0]$
m^2	$k(k-4)$	$k(k-2)$	$k(k+2)$	k^2-4	$k(k+4)$	k^2-4
Δ	k	$k+1$	$k+3$	$k+2$	$k+4$	$k+2$
twist	k	k	$k+2$	$k+2$	$k+4$	k

Table 2.1: $AdS_5 \times S^5$: KK modes contributing to exchange diagrams with four external superprimary modes s_k.

From the explicit expressions of the cubic vertices [30] one deduces two additional selection rules on the *twist* $\Delta - \ell$ of the field $\phi_{\mu_1 \ldots \mu_\ell}$ in order for the cubic vertex $s_{k_1} s_{k_2} \phi_{\mu_1 \ldots \mu_\ell}$ to be non-vanishing,

$$\Delta - \ell = k_1 + k_2 \pmod 2 \,, \qquad \Delta - \ell < k_1 + k_2 \,. \tag{2.5}$$

The selection rule on the parity of the twist can be understood as follows. In order for the cubic vertex $s_{k_1} s_{k_2} \phi_{\mu_1 \ldots \mu_\ell}$ to be non-zero, it is necessary for the "parent" vertex $s_{k_1} s_{k_2} s_{k_3}$ be non-zero, where s_{k_3} is the superprimary of which $\phi_{\mu_1 \ldots \mu_\ell}$ is a descendant. By $SU(4)$ selection rules, k_3 must have the same parity as $k_1 + k_2$. One then checks that all descendants of s_{k_3} that are allowed to couple to s_{k_1} and s_{k_2} by $SU(4)$ selection rules have the same twist parity as k_3. On the other hand, the selection rule $\langle \mathcal{O}^{k_1} \mathcal{O}^{k_2} \mathcal{O}^{k_1+k_2} \rangle$ is not fully explained by this kind of reasoning. To understand it, we first need to recall that the cubic vertices obtained in [29, 30] are cast in a "canonical form"

$$\int_{AdS_5} c_{ijk} \, \varphi_i \varphi_j \varphi_k \,, \tag{2.6}$$

by performing field redefinitions that eliminate vertices with spacetime derivatives. This is harmless so long as the twists of the three fields satisfy a strict triangular inequality, but subtle for the "extremal case" of one twist being equal to the sum of the other two [33]. For example, for the superprimaries, one finds that the cubic coupling $s_{k_1}s_{k_2}s_{k_1+k_2}$ is absent, in apparent contradiction with the fact that the in $\mathcal{N}=4$ SYM three-point function $\langle\mathcal{O}^{k_1}\mathcal{O}^{k_2}\mathcal{O}^{k_1+k_2}\rangle$ is certainly non-vanishing. One way to calculate $\langle\mathcal{O}^{k_1}\mathcal{O}^{k_2}\mathcal{O}^{k_3=k_1+k_2}\rangle$ is by analytic continuation in p_3 [29, 33]. One finds that while the coupling constant $c_{k_1k_2k_3}\sim(k_3-k_2-k_1)$, the requisite cubic contact Witten diagram diverges as $1/(k_3-k_1-k_2)$, so that their product yields the finite correct answer.[1] From this viewpoint, it is in fact *necessary* for the extremal coupling $c_{k_1k_2k_1+k_2}$ to vanish, or else one would find an infinite answer for the three-point function. This provides a rationale for the selection rule $\Delta-\ell<k_1+k_2$. When it is violated, the requisite three-point contact Wittten diagram diverges, so the corresponding coupling must vanish. We will see in Section 3.1.1, 3.1.2 of the next chapter that the selection rule has also a natural interpretation in Mellin space.

The requisite quartic vertices were obtained in [32]. The quartic terms in the effective action for the s_k fields contain up to four spacetime derivatives, but we argued in [1] that compatibility with the flat space limit requires that holographic correlators can get contributions from vertices with at most *two* derivatives. The argument is easiest to phrase in Mellin space and will be reviewed in Section 3.1.3. That is indeed the case in the handful of explicitly calculated examples [17, 18, 19, 20, 21, 22]. Our claim has been recently proven in full generality [35]. These authors have shown that the four-derivative terms effectively cancel out in all four-point correlators of one-half BPS operators, thanks to non-trivial group theoretic identities.

The rules of evaluation of Witten diagrams are entirely analogous to the

[1] If one wishes to work exactly at extremality $k_3=k_1+k_2$, one can understand the finite three-point function as arising from boundary terms that are thrown away by the field redefinition that brings the cubic vertex to the canonical non-derivative form [33]. One can rephrase this phenomenon as follows [34]: the field redefinition on the supergravity side (which throws away boundary terms) amounts to a redefinition of the dual operators that adds admixtures of multi-trace terms, $\mathcal{O}^p\to\mathcal{O}^p+1/N\sum_{k=2}^{p}c_k^p\,\mathcal{O}^{p-k}\mathcal{O}^k+\ldots$. The double-trace terms contribute to the extremal three point functions at leading large N order, but are subleading away from extremality. The operators dual to the redefined fields s_p (which have only non-derivative cubic couplings) are linear combinations of single and double-trace terms such all extremal three-point functions are zero, in agreement with the vanishing of the extremal three-point vertices $s_{p_1}s_{p_2}s_{p_1+p_2}$.

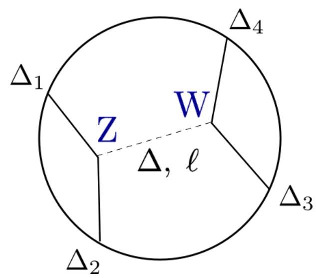

Figure 2.1: An exchange Witten diagram.

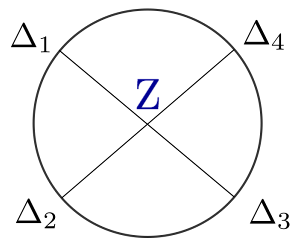

Figure 2.2: A contact Witten diagram.

ones for position space Feynman diagrams: we assign a bulk-to-bulk propagator $G_{BB}(z, w)$ to each internal line connecting two bulk vertices at positions z and w; and a bulk-to-boundary propagator $G_{B\partial}(z, \vec{x})$ to each external line connecting a bulk vertex at z and a boundary point $\vec{x}$. These propagators are Green's functions in AdS with appropriate boundary conditions. Finally, integrations over the bulk AdS space are performed for each interacting vertex point. The simplest connected Witten diagram is a contact diagram of external scalars with no derivatives in the quartic vertex (Figure 2.2). It is given by the integral of the product of four scalar bulk-to-boundary propagators integrated over the common bulk point,

$$\mathcal{A}_{\rm con}(\vec{x}_i) = \int_{AdS} dz\, G_{B\partial}(z, \vec{x}_1)\, G_{B\partial}(z, \vec{x}_2)\, G_{B\partial}(z, \vec{x}_3)\, G_{B\partial}(z, \vec{x}_4)\,. \tag{2.7}$$

Here, the scalar bulk-to-boundary propagator is [13][2],

$$G_{B\partial}(z,\vec{x}_i) = \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x}_i)^2}\right)^{\Delta_i} \tag{2.8}$$

where Δ_i is the conformal dimension of the ith boundary CFT operator. The integral can be evaluated in terms of derivatives of the dilogarithm function. It is useful to give it a name, defining the so-called D-functions as the four-point scalar contact diagrams with external dimensions Δ_i,

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4) \equiv \int_0^\infty \frac{dz_0}{z_0^{d+1}} \int d^d x \prod_{i=1}^{4} \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x}_i)^2}\right)^{\Delta_i} . \tag{2.9}$$

The other type of tree-level four-point diagrams are the exchange diagrams (Figure 2.1),

$$\mathcal{A}_{\rm ex}(\vec{x}_i) = \int_{AdS} dz dw G_{B\partial}(z,\vec{x}_1) G_{B\partial}(z,\vec{x}_2) G_{BB}(z,w) G_{B\partial}(w,\vec{x}_3) G_{B\partial}(w,\vec{x}_4) \tag{2.10}$$

Exchange diagrams are usually difficult to evaluate in closed form. In [36] a technique was invented that allows, when certain "truncation conditions" for the quantum numbers of the external and exchanged operators are met, to trade the propagator of an exchange diagram for a *finite* sum of contact vertices. In such cases, one is able to evaluate an exchange Witten diagram as a finite sum of D-functions. Fortunately, the spectrum and selection rules of IIB supergravity on $AdS_5 \times S^5$ are precisely such that all exchange diagrams obey the truncation conditions. We will exploit this fact in our position space method. The formulae for the requisite exchange diagrams have been collected in Appendix A.

In closing, let us mention that the qualitative features of eleven dimensional supergravity compactified on $AdS_7 \times S^4$ is the same: only contact diagrams with up to two derivatives contribute and all exchange Witten diagrams truncate. However for eleven dimensional supergravity compactified on $AdS_4 \times S^7$, the truncation condition is not met and the exchange Witten diagrams can not be evaluated as a finite sum of contact Witten diagrams.

[2]Note that we are using the *unnormalized* propagator, to avoid cluttering of several formulae. In a complete calculation, care must be taken to add the well-known normalization factors [15].

2.2 Some Superconformal Kinematics

In this section, let us discuss some superconformal kinematics of the four-point functions of one-half BPS operators. The discussion will be quite general: it applies to superconformal algebras in $d = 3, 4, 5, 6$ dimensions whose R-symmetry group is locally isomorphic to an $SO(n)$ group.[3] For such algebras, the one-half BPS operators $\mathcal{O}_k^{I_1 \dots I_k}$ are in the rank-k symmetric-traceless representation of $SO(n)$ and have quantized conformal dimension

$$\Delta = \epsilon k \ , \qquad \epsilon \equiv \frac{d}{2} - 1 \ . \tag{2.11}$$

The operators we discussed in the last section correspond to $d = 4$ $\mathcal{N} = 4$ and $n = 6$.

It is convenient to take care of the R-symmetry indices by contracting them with auxiliary null vectors t^I

$$\mathcal{O}_k(x, t) \equiv \mathcal{O}_k^{I_1 \dots I_k}(x)\, t_{I_1} \dots t_{I_k} \ , \qquad t^I t_I = 0 \ . \tag{2.12}$$

Then the four-point functions of such one-half BPS operators are index-free and depend on both the spacetime coordinates x_i and the internal R-symmetry coordinates t_i

$$G(x_i, t_i) \equiv \langle \mathcal{O}_{k_1}(x_1, t_1) \mathcal{O}_{k_2}(x_2, t_2) \mathcal{O}_{k_3}(x_3, t_3) \mathcal{O}_{k_4}(x_4, t_4) \rangle \ . \tag{2.13}$$

Define $t_{ij} \equiv t_i \cdot t_j$, the R-symmetry covariance and null property require that the t_i variables can only appear as sum of monomials $\prod_{i<j} (t_{ij})^{\gamma_{ij}}$ where the powers γ_{ij} are non-negative integers. Moreover, in order to have the correct scaling behavior when independently rescaling each null vector $t_i \to \zeta_i t_i$, the exponents γ_{ij} need to be further constrained by the condition $\sum_{i \neq j} \gamma_{ij} = k_j$. This set of constraints is solved with the following parameterization,

$$\begin{aligned}
\gamma_{12} &= -\frac{a}{2} + \frac{k_1 + k_2}{2} \ , & \gamma_{34} &= -\frac{a}{2} + \frac{k_3 + k_4}{2} \ , \\
\gamma_{23} &= -\frac{b}{2} + \frac{k_2 + k_3}{2} \ , & \gamma_{14} &= -\frac{b}{2} + \frac{k_1 + k_4}{2} \ , \\
\gamma_{13} &= -\frac{c}{2} + \frac{k_1 + k_3}{2} \ , & \gamma_{24} &= -\frac{c}{2} + \frac{k_2 + k_4}{2} \ ,
\end{aligned} \tag{2.14}$$

with the additional condition $a + b + c = k_1 + k_2 + k_3 + k_4$.

[3]They include 3d $OSp(4|\mathcal{N})$ ($\mathcal{N}$ even), 4d $(P)SU(2,2|\mathcal{N})$ with $\mathcal{N} = 2, 4$, 5d $F(4)$ and 6d $OSp(8^*|2\mathcal{N})$ with $\mathcal{N} = 1, 2$.

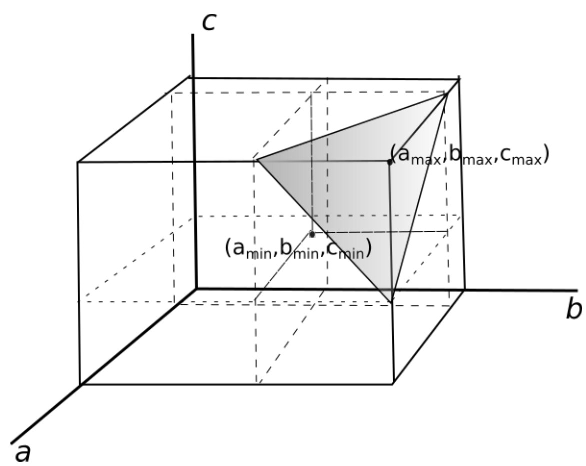

Figure 2.3: Solution to the γ_{ij} constraints.

We can assume $k_1 \geqslant k_2 \geqslant k_3 \geqslant k_4$ without loss of generality. It leaves us with two possibilities, namely,

$$k_1 + k_4 \leqslant k_2 + k_3 \quad \text{(case I)} \quad \text{and} \quad k_1 + k_4 > k_2 + k_3 \quad \text{(case II)} . \tag{2.15}$$

The inequality constraints $\gamma_{ij} \geqslant 0$ define in either case a cube inside the parameter space (a, b, c), as shown in Figure 2.3. The condition $a + b + c = k_1 + k_2 + k_3 + k_4$ further restricts the solution to be the equilateral triangle inside the cube shown shaded in the figure. We denote the coordinates of vertices of the cube closest and furthest from the origin as $(a_{\min}, b_{\min}, c_{\min})$ and $(a_{\max}, b_{\max}, c_{\max})$. Then

$$\begin{array}{ll}
a_{\max} = k_3 + k_4 \ , \ a_{\min} = k_3 - k_4 \ , & a_{\max} = k_3 + k_4 \ , \quad a_{\min} = k_1 - k_2 \ , \\
b_{\max} = k_1 + k_4 \ , \ a_{\min} = k_1 - k_4 \ , & b_{\max} = k_2 + k_3 \ , \quad a_{\min} = k_1 - k_4 \ , \\
c_{\max} = k_2 + k_4 \ , \ a_{\min} = k_2 - k_4 \ , & c_{\max} = k_2 + k_4 \ , \quad a_{\min} = k_1 - k_3 \ . \\
\text{(case I)} & \text{(case II)}
\end{array} \tag{2.16}$$

Let $2\mathcal{L}$ be the length of each side of the cube, we find

$$\mathcal{L} = k_4 \quad \text{(case I)} \ , \qquad \mathcal{L} = \frac{k_2 + k_3 + k_4 - k_1}{2} \quad \text{(case II)} \ . \tag{2.17}$$

It is clear from the parametrization (2.14) that γ_{ij} has lower bounds $\gamma_{ij} \geq \gamma_{ij}^0$. These γ_{ij}^0 are obtained by substituting the maximal values $(a_{\max}, b_{\max}, c_{\max})$,

$$
\begin{aligned}
\gamma^0_{12} &= \frac{k_1+k_2-k_3-k_4}{2} \ , \\
\gamma^0_{13} &= \frac{k_1+k_3-k_2-k_4}{2} \ , \\
\gamma^0_{34} &= \gamma^0_{24} = 0 \ , \\
\gamma^0_{14} &= 0 \quad (\text{case I}), \qquad \frac{k_1+k_4-k_2-k_3}{2} \ (\text{case II}) \ , \\
\gamma^0_{23} &= \frac{k_2+k_3-k_1-k_4}{2} \quad (\text{case I}), \qquad 0 \ (\text{case II}) \ .
\end{aligned} \tag{2.18}
$$

We now factor out the product $\prod_{i<j}\left(\frac{t_{ij}}{x_{ij}^{2\epsilon}}\right)^{\gamma^0_{ij}}$ from the correlator – each $\left(\frac{t_{ij}}{x_{ij}^{2\epsilon}}\right)^{k}$ is the two-point function of a weight-k one-half BPS operator. The object we obtain has the scaling behavior of a four-point function with equal weights $\mathcal{L}$. This behavior further motivates us to define

$$
G(x_i,t_i) = \prod_{i<j}\left(\frac{t_{ij}}{x_{ij}^{2\epsilon}}\right)^{\gamma^0_{ij}} \left(\frac{t_{12}t_{34}}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^{\mathcal{L}} \mathcal{G}(U,V;\sigma,\tau) \ . \tag{2.19}
$$

The outstanding factors take care of the covariance under the conformal and R-symmetry group, and the correlator is reduced into a function $\mathcal{G}(U,V;\sigma,\tau)$ depending on only four invariant variables. Here we used the usual conformal cross ratios

$$
U = \frac{(x_{12})^2(x_{34})^2}{(x_{13})^2(x_{24})^2} \ , \qquad V = \frac{(x_{14})^2(x_{23})^2}{(x_{13})^2(x_{24})^2} \tag{2.20}
$$

and analogously the R-symmetry cross ratios

$$
\sigma = \frac{(t_{13})(t_{24})}{(t_{12})(t_{34})} \ , \qquad \tau = \frac{(t_{14})(t_{23})}{(t_{12})(t_{34})} \ . \tag{2.21}
$$

It is not difficult to see that $\mathcal{G}(U,V;\sigma,\tau)$ is a polynomial of σ and τ with degree $\mathcal{L}$.

So far we have only required the correlator to be covariant under the bosonic part of the superconformal group. The fermionic generators impose further constraints on $\mathcal{G}(U,V;\sigma,\tau)$ in the form of a *superconformal Ward identity*. It is useful to make a change of variables

$$
\begin{aligned}
U &= \chi\chi' \ , \qquad & V &= (1-\chi)(1-\chi') \ , \\
\sigma &= \alpha\alpha' \ , \qquad & \tau &= (1-\alpha)(1-\alpha') \ .
\end{aligned} \tag{2.22}
$$

In terms of these variables, the superconformal Ward identity takes the universal form [37]

$$(\chi\partial_\chi - \epsilon\alpha\partial_\alpha)\mathcal{G}(\chi,\chi';\alpha,\alpha')\big|_{\alpha=1/\chi} = 0 \ . \tag{2.23}$$

2.3 An Efficient Position Space Method

As we reviewed in Section 2.1, a prerequisite of the traditional algorithm is the set of precise cubic and quartic vertices needed to compute the Witten diagrams. These vertices are usually obtained from the effective supergravity Lagrangian by perturbative expansion. However the devilishly complicated Kaluza-Klein supergravity makes such an expansion extremely difficult. General vertices for arbitrary Kaluza-Klein modes have only been explicitly obtained by Arutyunov and Frolov for the case of IIB supergravity on $AdS_5\times S^5$ [32]. Their final results for the quartic vertices filled a stunning 15 pages. For eleven dimensional supergravity compactified on $AdS_7 \times S^4$ and $AdS_4 \times S^7$, no such general result exists in the literature. Therefore these complicated vertices present a huge obstacle to implement the traditional method. In this section, we introduce an efficient method that circumvents the difficulty of obtaining vertices by exploiting symmetry.

The idea is to write the write full amplitude as a sum of exchange diagrams and contact diagrams, but parametrizing the vertices with undetermined coefficients. The spectra of IIB supergravity on $AdS_5 \times S^5$ and eleven dimensional supergravity on $AdS_7 \times S^4$ are such that all the exchange diagrams can be written as a finite sum of contact diagrams, *i.e.*, D-functions, making the whole amplitude a sum of D-functions. We then use the property of D-functions to decompose the amplitude into a basis of independent functions. The full amplitude is encoded into four rational coefficient functions. Imposing the superconformal Ward identity we find a large number of relations among the undetermined coefficients. Uniqueness of the maximally supersymmetric Lagrangian guarantees that all the coefficients in the ansatz can be fixed up to overall rescaling. Finally the overall constant can be determined by demanding that the OPE coefficient of an intermediate one-half BPS operator has the correct value. Alternatively, for $AdS_5 \times S^5$, we can fix the overall constant by comparing with the free field result after restricting the R-symmetry cross ratios to a special slice [38]. We emphasize that there is no guesswork anywhere. The position space method is *guaranteed* to give the same results as a direct supergravity calculation, but it is technically much simpler.

Let us now spell out the details. For simplicity, we focus on the equal-weight case where $k_i = k$. The ansatz is labelled by the integer k and takes the form

$$\mathcal{A}_k(U,V;\sigma,\tau) = \mathcal{A}_{k,\text{exchange}}(U,V;\sigma,\tau) + \mathcal{A}_{k,\text{contact}}(U,V;\sigma,\tau)\ . \tag{2.24}$$

Here we are working with the reduced correlator which depends only on the cross ratios and is obtained by stripping off the kinematic factor $\left(\frac{t_{12}\,t_{34}}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^k$ ($\epsilon = 1$ corresponds to $AdS_5 \times S^5$ and $\epsilon = 2$ corresponds to $AdS_7 \times S^4$). In this ansatz the exchange amplitude is summed over the three channels, related to one another by crossing,

$$\mathcal{A}_{k,\text{exchange}} = \mathcal{A}_{k,\text{s}-\text{ex}} + \mathcal{A}_{k,\text{t}-\text{ex}} + \mathcal{A}_{k,\text{u}-\text{ex}}\ , \tag{2.25}$$

$$\begin{aligned} \mathcal{A}_{k,\text{t}-\text{ex}}(U,V;\sigma,\tau) =&\ \left(\frac{U^2\tau}{V^2}\right)^k \mathcal{A}_{k,\text{s}-\text{ex}}(V,U;\sigma/\tau,1/\tau)\ , \\ \mathcal{A}_{k,\text{u}-\text{ex}}(U,V;\sigma,\tau) =&\ \left(U^2\sigma\right)^k \mathcal{A}_{k,\text{s}-\text{ex}}(1/U,V/U;1/\sigma,\tau/\sigma)\ . \end{aligned} \tag{2.26}$$

The s-channel exchange amplitude $\mathcal{A}_{k,\text{s}-\text{ex}}$ is given by the sum of all s-channel exchange Witten diagrams compatible with the selection rules for the cubic vertices. Schematically,

$$\mathcal{A}_{k,\text{s}-\text{ex}} = \sum_X \lambda_X Y_{R_X}(\sigma,\tau)\mathcal{E}_X(U,V)\ . \tag{2.27}$$

Here the label X runs over the exchanged fields, $\mathcal{E}_X$ denotes the corresponding exchange Witten diagram, $Y_{R_X}(\sigma,\tau)$ is the polynomial associated with the irreducible representation R_X of the field X and finally λ_X are the unknown coefficients to be determined. The expressions for the R-symmetry polynomials $Y_{R_X}(\sigma,\tau)$ will be given shortly when we discuss concrete examples. The expressions for the exchange Witten diagrams have been given in Appendix A.

The discussion of contribution from contact diagrams should distinguish two difference cases, as we explain in Appendix B. When $\Delta_i = d$, we should include in the ansatz both the contributions from contact vertices with no derivatives and with two-derivatives. When $\Delta_i \neq d$, the zero-derivative contribution can be absorbed into the two-derivative contribution by redefining the parameters in the ansatz. Because of crossing symmetry, it is also convenient to split the two-derivative contribution into channels. The full contact vertex is the crossing symmetrization of the following s-channel contribution,

$$S_{\alpha_1\alpha_2\alpha_3\alpha_4}\int_{AdS_{d+1}} dX\ \frac{1}{16}(\nabla s^{\alpha_1}\nabla s^{\alpha_2}s^{\alpha_3}s^{\alpha_4}+s^{\alpha_3}s^{\alpha_4}\nabla s^{\alpha_1}\nabla s^{\alpha_2})\,, \tag{2.28}$$

where s^α is the scalar field dual to the one-half BPS operator and α_i collectively denote the indices of the symmetric traceless representation. $S_{\alpha_1\alpha_2\alpha_3\alpha_4}$ is an unspecified tensor symmetric under $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$. When contracted with the R-symmetry null vectors the above vertices lead to the following contribution to the s-channel contact amplitude,

$$\mathcal{A}_{k,\mathrm{s-cont}} \propto \sum_{0\leq a+b\leq k} c_{ab}\sigma^a\tau^b U^{\epsilon k}(\bar{D}_{\epsilon k,\epsilon k,\epsilon k,\epsilon k}-\frac{2(2\epsilon k-\epsilon-1)}{\epsilon^2k^2}U\bar{D}_{\epsilon k,\epsilon k,\epsilon k+1,\epsilon k+1})\,. \tag{2.29}$$

Here we have used the so-called $\bar{D}$-functions[4] defined by stripping off some kinematic factors from the D-functions,

$$\frac{\prod_{i=1}^4\Gamma(\Delta_i)}{\Gamma(\Sigma-\frac{1}{2}d)}\frac{2}{\pi^{\frac{d}{2}}}D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1,x_2,x_3,x_4)=\frac{r_{14}^{\Sigma-\Delta_1-\Delta_4}r_{34}^{\Sigma-\Delta_3-\Delta_4}}{r_{13}^{\Sigma-\Delta_4}r_{24}^{\Delta_2}}\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(U,V)\,, \tag{2.31}$$

with $\Sigma=\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3+\Delta_4)$. The coefficients c_{ab} in (2.29) are symmetric thanks to the exchange symmetry $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$. When $\Delta_i = d$, we need to also include the zero-derivative contribution

$$\sum_{0\leq a+b\leq k} c'_{ab}\sigma^a\tau^b U^{\epsilon k}\bar{D}_{\epsilon k,\epsilon k,\epsilon k,\epsilon k}\,,\qquad c'_{ab}=c'_{ba}\,. \tag{2.32}$$

[4]We emphasize that $\bar{D}$-functions are independent of the spacetime dimension d. This is clearest from their Mellin-Barnes representation (See Section 3.1 for more details),

$$\begin{aligned}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} = \int \frac{ds}{2}\frac{dt}{2}U^{\frac{s}{2}}V^{\frac{t}{2}}&\Gamma[-\frac{s}{2}]\Gamma[-\frac{s}{2}+\frac{\Delta_3+\Delta_4-\Delta_1-\Delta_2}{2}]\\ &\times\Gamma[-\frac{t}{2}]\Gamma[-\frac{t}{2}+\frac{\Delta_1+\Delta_4-\Delta_2-\Delta_3}{2}]\\ &\times\Gamma[\Delta_2+\frac{s+t}{2}]\Gamma[\frac{s+t}{2}+\frac{\Delta_1+\Delta_2+\Delta_3-\Delta_4}{2}]\,,\end{aligned} \tag{2.30}$$

where d completely drops out.

The crossed channel contributions $\mathcal{A}_{k,\mathrm{t-cont}}$ and $\mathcal{A}_{k,\mathrm{u-cont}}$ can then be obtained from $\mathcal{A}_{k,\mathrm{s-cont}}$ using the crossing relation (2.26).

Putting all these pieces together, we now have an anstaz $\mathcal{A}_k(U,V;\sigma,\tau)$ of the four-point function as a finite sum of $\bar{D}$-functions. It has polynomial dependence on σ and τ and contains linearly all the unspecified coefficients λ_X, c_{ab}, c'_{ab}. These coefficients must to be fine-tuned in order to satisfy the superconformal Ward identity (2.23),

$$(\chi'\partial_{\chi'}-\epsilon\alpha'\partial_{\alpha'})\mathcal{G}_k(\chi,\chi';\alpha,\alpha')\big|_{\alpha'\to1/\chi'}=0\,. \tag{2.33}$$

The ansatz $\mathcal{A}_k$ is not yet in a form such the superconformal Ward identity can be conveniently exploited. Fortunately, all $\bar{D}$-functions that appear in the ansatz can be reached from the basic $\bar{D}$-function $\bar{D}_{1111}$ with the repetitive use of six differential operators,

$$\begin{aligned}
\bar{D}_{\Delta_1+1,\Delta_2+1,\Delta_3,\Delta_4}&=\mathcal{D}_{12}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}:=-\partial_U\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\,,\\
\bar{D}_{\Delta_1,\Delta_2,\Delta_3+1,\Delta_4+1}&=\mathcal{D}_{34}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}:=(\Delta_3+\Delta_4-\Sigma-U\partial_U)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\,,\\
\bar{D}_{\Delta_1,\Delta_2+1,\Delta_3+1,\Delta_4}&=\mathcal{D}_{23}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}:=-\partial_V\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\,,\\
\bar{D}_{\Delta_1+1,\Delta_2,\Delta_3,\Delta_4+1}&=\mathcal{D}_{14}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}:=(\Delta_1+\Delta_4-\Sigma-V\partial_V)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\,,\\
\bar{D}_{\Delta_1,\Delta_2+1,\Delta_3,\Delta_4+1}&=\mathcal{D}_{24}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}:=(\Delta_2+U\partial_U+V\partial_V)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\,,\\
\bar{D}_{\Delta_1+1,\Delta_2,\Delta_3+1,\Delta_4}&=\mathcal{D}_{13}\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}:=(\Sigma-\Delta_4+U\partial_U+V\partial_V)\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}\,.
\end{aligned} \tag{2.34}$$

The special function $\bar{D}_{1111}$ is in fact the familiar scalar one-loop box integral in four dimensions and will be denoted as Φ from now on. It has a well-known representation in terms of dilogarithms,

$$\Phi(\chi,\chi')=\frac{1}{\chi-\chi'}\left(\log(\chi\chi')\log(\frac{1-\chi}{1-\chi'})+2\mathrm{Li}(\chi)-2\mathrm{Li}(\chi')\right)\,, \tag{2.35}$$

and enjoys the following beautiful differential recursion relations [39]

$$\begin{aligned}
\partial_\chi\Phi&=-\frac{\Phi}{\chi-\chi'}-\frac{\ln[(-1+\chi)(-1+\chi')]}{\chi(\chi-\chi')}+\frac{\ln[\chi\chi']}{(-1+\chi)(\chi-\chi')}\,,\\
\partial_{\chi'}\Phi&=\frac{\Phi}{\chi-\chi'}+\frac{\ln[(-1+\chi)(-1+\chi')]}{\chi'(\chi-\chi')}-\frac{\ln[\chi\chi']}{(-1+\chi')(\chi-\chi')}\,.
\end{aligned} \tag{2.36}$$

Using the above properties of $\bar{D}$-functions, we can unambiguously decompose the supergravity ansatz into a basis spanned by Φ, $\log U$, $\log V$ and 1,

$$\mathcal{A}_k(\chi,\chi';\alpha,\alpha')=R_\Phi\Phi(\chi,\chi')+R_{\log U}\log U+R_{\log V}\log V+R_1\,, \tag{2.37}$$

where the four coefficients functions R_Φ, $R_{\log U}$, $R_{\log V}$ and R_1 are rational functions of χ, χ' and polynomials of α, α'. This decomposition makes it straightforward to enforce the superconformal Ward identity (2.23) on $\mathcal{A}_k$. Upon acting on $\mathcal{A}_k(\chi,\chi';\alpha,\alpha')$ with the differential operator $(\chi'\partial_{\chi'}-2\alpha'\partial_{\alpha'})$ from (2.23) and setting $\alpha'=1/\chi'$, a new set of coefficient functions $\widetilde{R}_\Phi$, $\widetilde{R}_{\log U}$, $\widetilde{R}_{\log V}$ $\widetilde{R}_1$ are generated from R_Φ, $R_{\log U}$, $R_{\log V}$, R_1 with the help of the differential recursion relation of Φ. The superconformal Ward identity then dictates the following conditions

$$\begin{aligned}
&\widetilde{R}_\Phi(\chi,\chi';\alpha,1/\chi')=0\ ,\\
&\widetilde{R}_{\log U}(\chi,\chi';\alpha,1/\chi')=0\ ,\\
&\widetilde{R}_{\log V}(\chi,\chi';\alpha,1/\chi')=0\ ,\\
&\widetilde{R}_1(\chi,\chi';\alpha,1/\chi')=0\ ,
\end{aligned}\tag{2.38}$$

which imply a large set of linear equations for the unknown coefficients. This set of equations is constraining enough to fix all relative coefficients up to an overall constant. That the overall constant should remain undetermined is inevitable because the condition (2.23) is homogeneous. To fix it, we demand that the OPE coefficient of the intermediate one-half BPS operator $\mathcal{O}^{(2)}$ has the correct value. The details of this calculation are discussed in Appendix B of [5]. For $AdS_5\times S^5$ where the dual 4d $\mathcal{N}=4$ SYM theory has a marginal coupling, we can also use the free theory limit.

Now let us apply this method to compute some holographic four-point functions.

2.3.1 Sample Computations

$k=2$ **for** $AdS_5\times S^5$

In the s-channel, we know from Table 2.1 and the twist cut-off $\tau<4$ that there are only three fields which can be exchanged: there is an exchange of scalar with dimension two and in the representation $[0,2,0]$,

$$\mathcal{A}_{\rm scalar}=\frac{1}{8}\pi^2\lambda_s U(3\sigma+3\tau-1)\bar{D}_{1122}\,,\tag{2.39}$$

a vector of dimension three in the representation $[1,0,1]$

$$\mathcal{A}_{\rm vector}=\frac{3}{8}\pi^2\lambda_v U(\sigma-\tau)\left(\bar{D}_{1223}-\bar{D}_{2123}+\bar{D}_{2132}-V\bar{D}_{1232}\right)\ ,\tag{2.40}$$

and a massless symmetric graviton in the singlet representation,

$$\mathcal{A}_{\text{graviton}} = \frac{1}{3}(-2)\pi^2\lambda_g U \left(2\bar{D}_{1122} - 3\left(\bar{D}_{2123} + \bar{D}_{2132} - \bar{D}_{3133}\right)\right) \ . \tag{2.41}$$

In the above expressions we have used the formulae for exchange Witten diagrams from Appendix A and multiplied with the explicit expression of R-symmetry polynomials Y_{00}, Y_{11}, Y_{10}. These R-symmetry polynomials Y_{mn} are derived in [38], and read

$$Y_{nm}(\alpha, \bar{\alpha}) = \frac{P_n(\alpha)P_m(\bar{\alpha}) - P_m(\alpha)P_n(\bar{\alpha})}{\alpha - \bar{\alpha}} \ , \tag{2.42}$$

The constants λ_s, λ_v and λ_g are undetermined parameters.

For the contact diagram, we only need to consider two-derivative vertices. The most general contribution is as follows (only in the s-channel, as we will sum over the channels in the next step),

$$\mathcal{A}_{2,\text{s}-\text{con}} = -\Bigg(\sum_{0\le a+b\le 2} c_{ab}\sigma^a\tau^b\Bigg)2\pi^2U^2(-2\bar{D}_{2222} + \bar{D}_{2233} + U\bar{D}_{3322}) \tag{2.43}$$

where $c_{ab} = c_{ba}$ because the s-channel is symmetric under the exchange of 1 and 2.

Being a sum of $\bar{D}$-functions, $\mathcal{A}_2$ can be systematically decomposed into Φ, $\ln U$, $\ln V$ and the rational part. For example, the coefficient function of Φ is of the form

$$R_\phi(z, \bar{z}, \alpha, \bar{\alpha}) = \frac{T(z, \bar{z}, \alpha, \bar{\alpha})}{(z - \bar{z})^6} \tag{2.44}$$

where the numerator $T(z, \bar{z}, \alpha, \bar{\alpha})$ a polynomial of degree 2 in α, $\bar{\alpha}$ and of degree 12 in z and $\bar{z}$. The superconformal Ward identity then requires $T(z, \bar{z}; \alpha, 1/\bar{z}) = 0$ and reduces to a set of homogenous linear equations. Their solution is

$$\begin{gathered} \lambda_s = \xi, \qquad \lambda_v = -\frac{1}{2}\xi, \qquad \lambda_g = \frac{3}{16}\xi \ , \\ c_{00} = \frac{3}{32}\xi, \quad c_{01} = -\frac{3}{8}\xi, \quad c_{02} = \frac{3}{32}\xi, \quad c_{11} = -\frac{3}{16}\xi \ , \end{gathered} \tag{2.45}$$

where ξ is an arbitrary overall constant. We then compute "twisted" correlator

$$\mathcal{A}_2(\alpha, 1/\bar{z}, z, \bar{z}) = -\frac{3\pi^2\zeta\left(\alpha^2z^2 - 2\alpha z^2 + 2\alpha z - z\right)}{8N^2(z-1)} \ , \tag{2.46}$$

and compare it to the free field result

$$\mathcal{G}_{\text{free,conn}}(\alpha, 1/\bar{z}, z, \bar{z}) = -\frac{4\,(\alpha^2 z^2 - 2\alpha z^2 + 2\alpha z - z)}{N^2(z-1)}\ . \tag{2.47}$$

The functional agreement of the two expressions provides a consistency check, and fixes the value of the last undetermined constant,

$$\xi = \frac{32}{3N^2\pi^2}\ . \tag{2.48}$$

The final answer agrees with the result in the literature [17].

$k = 4$ **for** $AdS_5 \times S^5$

The computation of $k = 3$ is similar to that of $k = 2$. Therefore let us move onto the next example of $k = 4$ where the computation is slightly different. The $k = 4$ case is special in that we cannot use two-derivative contact vertices to absorb the contribution of zero-derivative ones by redefinition the parameters. Therefore in this case we must include both types of contributions in the ansatz. The s-channel ansatz is given by

$$\begin{aligned}\mathcal{A}_{4,\text{s}-\text{channel}} = {} & \lambda_{s_2}\mathcal{A}_{s_2} + \lambda_{A_2}\mathcal{A}_{A_2} + \lambda_{\varphi_2}\mathcal{A}_{\varphi_2} \\ & +\lambda_{s_4}\mathcal{A}_{s_4} + \lambda_{A_4}\mathcal{A}_{A_4} + \lambda_{\varphi_4}\mathcal{A}_{\varphi_4} + \lambda_{C_4}\mathcal{A}_{C_4} + \lambda_{\phi_4}\mathcal{A}_{\phi_4} \\ & +\lambda_{s_6}\mathcal{A}_{s_6} + \lambda_{A_6}\mathcal{A}_{A_6} + \lambda_{\varphi_6}\mathcal{A}_{\varphi_6} \\ & +\mathcal{A}_{\text{contact}}\end{aligned} \tag{2.49}$$

where

$$\begin{aligned}\mathcal{A}_{4,\text{s}-\text{cont}} = {} & \Bigg(\sum_{0\le a+b\le 4} c_{ab}\sigma^a\tau^b\Bigg)\frac{5\pi^2U^2}{216}(4\bar{D}_{4444} - 3U\bar{D}_{5544}) \\ & + \Bigg(\sum_{0\le a+b\le 4} c'_{ab}\sigma^a\tau^b\Bigg)\frac{5\pi^2U^2}{108}\bar{D}_{4444}.\end{aligned} \tag{2.50}$$

The superconformal Ward identity is expected not to fix all the coefficients because we know certain crossing symmetric choice of the two-derivative contact coupling will give a zero contribution. As it turned out, all these unsolved coefficients are multiplied by a common factor

$$-8\bar{D}_{4444} + \bar{D}_{4455} + \bar{D}_{4545} + V\bar{D}_{4554} + \bar{D}_{5445} + \bar{D}_{5454} + U\bar{D}_{5544} \tag{2.51}$$

which is identically zero by $\bar{D}$-identities. These coefficients can be set to zero at our convenience.

The solution is

$$\begin{aligned}
&\lambda_{s_2} = \frac{3456}{\pi^2 N^2}, \quad \lambda_{A_2} = -\frac{384}{\pi^2 N^2}, \quad \lambda_{\varphi_2} = \frac{18}{\pi^2 N^2}, \\
&\lambda_{s_4} = \frac{18432}{5\pi^2 N^2}, \quad \lambda_{A_4} = -\frac{1728}{5\pi^2 N^2}, \quad \lambda_{\varphi_4} = \frac{288}{25\pi^2 N^2}, \\
&\lambda_{C_4} = -\frac{192}{25\pi^2 N^2}, \quad \lambda_{\phi_4} = \frac{576}{5\pi^2 N^2}, \\
&\lambda_{s_6} = \frac{15552}{35\pi^2 N^2}, \quad \lambda_{A_6} = -\frac{5184}{175\pi^2 N^2}, \quad \lambda_{\varphi_6} = \frac{18}{25\pi^2 N^2}, \\
&c_{12} = \frac{1728}{5\pi^2 N^2}, c_{13} = \frac{576}{5\pi^2 N^2}, c_{22} = \frac{2304}{5\pi^2 N^2}, \\
&c'_{04} = \frac{216}{5\pi^2 N^2}, c'_{12} = -\frac{16848}{5\pi^2 N^2}, c'_{13} \to \frac{576}{5\pi^2 N^2}, c'_{22} = -\frac{8928}{5\pi^2 N^2}
\end{aligned} \tag{2.52}$$

with all the other unlisted coefficients being zero.

$k = 2$ for $AdS_7 \times S^4$

We now use the position space method to compute four-point functions for eleven dimensional supergravity on $AdS_7 \times S^4$. The cubic selection rules are similar: exchanged fields are subject to R-symmetry selections as well as a twist cut-off. We also collect below the requisite R-symmetry polynomials [38] for $k = 2, 3$:

$$\begin{aligned}
Y_{00} =&1 , \\
Y_{10} =&\sigma - \tau , \\
Y_{11} =&\sigma - \tau - \frac{2}{d} , \\
Y_{20} =&\sigma^2 + \tau^2 - 2\sigma\tau - \frac{2}{d-2}(\sigma + \tau) + \frac{2}{(d-2)(d-1)} , \\
Y_{21} =&\sigma^2 - \tau^2 - \frac{4}{d+2}(\sigma - \tau) , \\
Y_{22} =&\sigma^2 + \tau^2 + 4\sigma\tau - \frac{8}{d+4}(\sigma + \tau) + \frac{8}{(d+2)(d+4)} .
\end{aligned} \tag{2.53}$$

In the above expressions $d \equiv 5$. The $USp(4)$ Dynkin labels $[2(a-b), 2b]$ are related to the labels (m, n) in Y_{nm} via $n = a$, $m = b$.

field	ℓ	R-irrep	m^2	Δ	$k=2$	$k=3$	$k=4$
$\varphi_{\mu\nu,k}$	2	$[0,k-2]$	$4(k-2)(k+1)$	$2k+2$	6	8	10
$A_{\mu,k}$	1	$[2,k-2]$	$4k(k-2)$	$2k+1$	5	7	9
$C_{\mu,k}$	1	$[2,k-4]$	$4(k-1)(k+1)$	$2k+3$	-	-	11
s_k	0	$[0,k]$	$4k(k-3)$	$2k$	4	6	8
t_k	0	$[0,k-4]$	$4(k-1)(k+2)$	$2k+4$	-	-	12
r_k	0	$[4,k-4]$	$4(k-2)(k+1)$	$2k+2$	-	-	10

Table 2.2: $AdS_7 \times S^4$: KK modes contributing to exchange diagrams allowed by R-symmetry selection rules.

We start with the $k=2$ one-half BPS operator $O^{(2)}$ which sits in the same short supermultiplet as the stress tensor. Its four-point function was first calculated in [25] and we will reproduce their result. By the two selection rules of cubic vertices the allowed exchanges are identified to be all the fields that belong to the $k=2$ family in the following Table 2.2. Explicitly, the exchange Witten diagrams in the s-channel contribute

$$\mathcal{A}_{2,\,\text{s-exchange}} = Y_{11}\lambda_{s_2}\mathcal{E}_{s_2} + Y_{10}\lambda_{A_2}\mathcal{E}_{A_2} + \lambda_{\varphi_2}\mathcal{E}_{\varphi_2}\ . \tag{2.54}$$

As was discussed above, the contribution of contact Witten diagrams can be split into channels and then cross-symmetrized. Moreover, because $\Delta_i \neq d$ we can absorb the contribution of the zero-derivative terms into the two-derivative terms. Hence we have the following s-channel ansatz for the contact contributions,

$$\mathcal{A}_{2,\,\text{s-contact}} = \sum_{0\leq a+b\leq 2} c_{ab}\tau^a\sigma^b \frac{\pi^3 U^4}{432}(8\bar{D}_{4444} - 5U\bar{D}_{5544}) \tag{2.55}$$

where $c_{ab} = c_{ba}$ follows from symmetry under exchanging operators 1 and 2. The total amplitude $\mathcal{A}_2$ is obtained from cross-symmetrizing the above s-channel amplitude,

$$\begin{aligned}\mathcal{A}_2(U,V;\sigma,\tau) &= \mathcal{A}_{2,s}(U,V;\sigma,\tau) + \left(\frac{U^2\tau}{V^2}\right)^2 \mathcal{A}_{2,s}(V,U;\sigma/\tau,1/\tau) \\ &\quad + \left(U^2\sigma\right)^2 \mathcal{A}_{2,s}(1/U,V/U;1/\sigma,\tau/\sigma)\ , \\ \mathcal{A}_{2,s}(U,V;\sigma,\tau) &= \mathcal{A}_{2,\,\text{s-exchange}}(U,V;\sigma,\tau) + \mathcal{A}_{2,\,\text{s-contact}}(U,V;\sigma,\tau)\ .\end{aligned} \tag{2.56}$$

Decomposing this ansatz into the basis of functions Φ, $\log U$, $\log V$ and 1 and enforcing the superconformal Ward identity (2.38), we find enough constraints to fix all the coefficients up to an overall factor ξ,

$$\begin{aligned}&\lambda_{s_2}=\xi, \quad \lambda_{A_2}=-\frac{1}{9}\xi, \quad \lambda_{\varphi_2}=\frac{1}{576}\xi, \\ &c_{00}=\frac{1}{36}\xi, \quad c_{01}=-\frac{1}{9}\xi, \quad c_{02}=\frac{1}{36}\xi, \quad c_{11}=-\frac{1}{12}\xi\ .\end{aligned} \tag{2.57}$$

The last coefficient can be determined by demanding that the relevant term in the OPE is compatible with the known value of the three-point coupling $\langle\mathcal{O}^2\mathcal{O}^2\mathcal{O}^2\rangle$. The details of this computation are discussed in Appendix B of [5]. The result is

$$\xi=\frac{15552}{\pi^3 n^3}\ . \tag{2.58}$$

$k=3$ **for** $AdS_7\times S^4$

The calculation of the $k=3$ correlator for $AdS_7\times S^4$ is similar to $k=4$ for $AdS_5\times S^5$. The allowed exchanges include the three component fields of the $k=2$ family in Table 2.2 and all other fields of the $k=4$ family except for the field t_4. This field is ruled out because it has twist 12, which violates the twist upper bound. The $k=3$ family, on the other hand, is absent because of the R-symmetry selection rule. For the contact diagrams we notice that in this case the conformal dimension of the external operators coincides with the boundary spacetime dimension $\Delta=d$. As was discussed in the Appendix B, the zero-derivative contribution can no longer be reabsorbed into the two-derivative contribution. We need to include in our ansatz both set of parameters for the quartic vertices, even if this will lead to some (harmless) ambiguities in fixing the coefficients of contact vertices when we use the superconformal Ward identity.

The s-channel ansatz is again given by an exchange part $\mathcal{A}_{3,\,\text{s-exchange}}$ and a contact part $\mathcal{A}_{3,\text{s-contact}}$

$$\begin{aligned}\mathcal{A}_{3,\,\text{s-exchange}}=&Y_{11}\lambda_{s_2}\mathcal{E}_{s_2}+Y_{10}\lambda_{A_2}\mathcal{E}_{A_2}+Y_{00}\lambda_{\varphi_2}\mathcal{E}_{\varphi_2}\\ &+Y_{22}\lambda_{s_4}\mathcal{E}_{s_4}+Y_{21}\lambda_{A_4}\mathcal{E}_{A_4}+Y_{11}\lambda_{\varphi_4}\mathcal{E}_{\varphi_4}+Y_{20}\lambda_{r_4}\mathcal{E}_{r_4}+Y_{10}\lambda_{C_4}\mathcal{E}_{C_4}\ ,\end{aligned} \tag{2.59}$$

$$
\begin{aligned}
\mathcal{A}_{3,\text{s}-\text{contact}} = & \sum_{0\leq a+b\leq 3} c_{ab}\tau^a\sigma^b \frac{7\pi^3 U^6}{72000}(2\bar{D}_{6666} - U\bar{D}_{7766}) \\
& + \sum_{0\leq a+b\leq 3} c'_{ab}\tau^a\sigma^b \frac{7\pi^3 U^6}{36000}\bar{D}_{6666}
\end{aligned}
\tag{2.60}
$$

where the coefficients $c_{ab} = c_{ba}$, $c'_{ab} = c'_{ba}$ are symmetric. The total amplitude ansatz is obtained by further including the t-channel and u-channel contributions which are obtained from the above s-channel contribution by crossing. Due to the ambiguity in the parameterizing of contact vertices, the superconformal Ward identity fixes only the coefficients of the exchange diagrams, leaving a subset of c_{ab}, c'_{ab} unfixed,

$$
\begin{aligned}
& \lambda_{s_2} = \xi, \quad \lambda_{A_2} = -\frac{3}{50}\xi, \quad \lambda_{\varphi_2} = \frac{1}{3600}\xi, \\
& \lambda_{s_4} = \frac{224}{135}\xi, \quad \lambda_{A_4} = -\frac{8}{105}\xi, \quad \lambda_{\varphi_4} = \frac{1}{5040}\xi, \quad \lambda_{C_4} = -\frac{1}{4900}\xi, \quad \lambda_{r_4} = \frac{16}{945}\xi, \\
& c_{12} = \frac{1}{140}\xi, \quad c'_{03} = \frac{1}{630}\xi, \quad c'_{11} = -\frac{17}{630}\xi, \quad c'_{12} = -\frac{13}{1260}\xi\ .
\end{aligned}
\tag{2.61}
$$

These unfixed coefficients are actually redundant: the corresponding expressions are proportional to a sum of $\bar{D}$-functions which is zero in disguise, thanks to $\bar{D}$-function identities. We can then set them to zero (or to any convenient value). The last coefficient is fixed by enforcing the correct value of the OPE coefficient of $\mathcal{O}^{(2)}$, which gives

$$
\xi = \frac{1080000}{n^3\pi^3}\ . \tag{2.62}
$$

2.4 Conclusion

The position space method we introduced in this chapter circumvents many difficulties in the traditional algorithm. In particular, it is no longer necessary to input the precise effective Lagrangian from which the requisite vertices in the traditional algorithm are obtained. The contribution of each diagram is instead efficiently fixed by exploiting the superconformal symmetry. Using this improved method, we not only reproduced the $k = 2, 3, 4$ results for

$AdS_5 \times S^5$ [17, 18, 19], but also computed the new case $k = 5$ [4], confirming a conjectural result in [23]. For $AdS_7 \times S^4$ we extended the known $k = 2$ result [25] by computing two more cases with $k = 3$ and $k = 4$ [5]. Though streamlined and much simpler than the traditional algorithm, the position space method has its own shortcomings. This method also encounters intractable computational difficulty as the weighs of the external operators are increased. The difficulty is associated with the proliferation of exchange diagrams, and is also related to the increasing computing time of decomposing D-functions with large conformal dimensions. Moreover, because the position space method leverages the special property that exchange Witten diagrams on certain background can be expressed as finitely contact diagrams, the range of its applicability is limited. The position space method cannot be extended to other physically interesting theories (such as k=1 ABJM which is dual to eleven dimensional supergravity on $AdS_4 \times S^7$) where this feature is absent.

Nevertheless, the most important part of this method is the idea of bootstrapping holographic correlators. This idea will be inherited, but we will need a better formalism in which we can have more analytic control. This formalism is the the Mellin representation formalism that we will discuss in the next Chapter. In this thesis, we will combine this formalism with superconformal symmetry, and formulate two complimentary methods to overcome the difficulties mentioned above.

Chapter 3

Mellin Representation Formalism

In this chapter, we give a detailed exposition of the Melin representation formalism [40, 41]. In particular we will emphasize the holographic use of this formalism which is most relevant to application in this thesis.[1] We also make an extension of this formalism to conformal field theories admitting a conformal boundary or a defect (interface) [3].

3.1 Mellin Formalism for Conformal Field Theories

We consider a general correlation function of n scalar operators with conformal dimensions Δ_i. Conformal symmetry restricts its form to be

$$G_{\Delta_1,\ldots,\Delta_n}(x_1,\ldots,x_n) = \prod_{i<j}(x_{ij}^2)^{-\delta_{ij}^0}\mathcal{G}(\xi_r)\,, \tag{3.1}$$

where ξ_r are the conformally invariant cross ratios constructed from x_{ij}^2,

$$\frac{(x_i-x_j)^2(x_k-x_l)^2}{(x_i-x_l)^2(x_k-x_j)^2}\,. \tag{3.2}$$

[1]See [41, 42, 43, 44, 45, 46, 47, 48, 49, 1, 50, 3, 4, 51, 5, 6] for applications at tree level and [45, 52, 53] at loop level.

Requiring that the correlator transforms with appropriate weights under conformal transformations, one finds the constraints

$$\sum_{j\neq i}\delta_{ij}^{0}=\Delta_i\ . \tag{3.3}$$

The number of independent cross ratios in a d-dimensional spacetime is given by

$$\begin{aligned} n<d+1: &\quad \frac{1}{2}n(n-3)\ ,\\ n\geqslant d+1: &\quad nd-\frac{1}{2}(d+1)(d+2)\ , \end{aligned} \tag{3.4}$$

as seen from a simple counting argument. We have a configuration space of n points which is nd-dimensional, while the dimension of the conformal group $SO(d+1,1)$ is $\frac{1}{2}(d+1)(d+2)$. For sufficiently large n, the difference of the two gives the number of free parameters unfixed by the conformal symmetry, as in the second line of (3.4). However this is incorrect for $n<d+1$ because we have overlooked a nontrivial stability group. To see this, we first use a conformal transformation to send two of the n points to the origin and the infinity. If $n<d+1$, the remaining $n-2$ points will define a hyperplane and the stability group is the rotation group $SO(d+2-n)$ perpendicular to the hyperplane. After adding back the dimension of the stability group we get the first line of the counting. To phrase it differently, when the spacetime dimension d is high enough, there are always $\frac{1}{2}n(n-3)$ conformal cross ratios, independent of the spacetime dimension. But when $n\geq d+1$ there exist nontrivial algebraic relations among the $\frac{1}{2}n(n-3)$ conformal cross ratios.

The constraints (3.3) admit $\frac{1}{2}n(n-3)$ solutions, in correspondence with the $\frac{1}{2}n(n-3)$ cross ratios (ignoring the algebraic relations that exist for small n). Mack [40] suggested instead of taking δ_{ij}^{0} to be fixed, we should view them as variables δ_{ij} satisfying the same constraints,

$$\delta_{ij}=\delta_{ji}\,,\quad \sum_{j}\delta_{ij}=\Delta_i\,, \tag{3.5}$$

and write the correlator as an integral transform with respect to these variables. More precisely, one defines the following (inverse) Mellin transform for the *connected*[2] part of the correlator,

$$G^{\text{conn}}_{\Delta_1,\dots,\Delta_n}(x_1,\dots,x_n) = \int [d\delta_{ij}] M(\delta_{ij}) \prod_{i<j} (x_{ij}^2)^{-\delta_{ij}} \tag{3.6}$$

The integration is performed with respect to the $\frac{1}{2}n(n-3)$ independent variables along the imaginary axis. We will be more specific about the integration in a moment. The correlator $\mathcal{G}(\xi_r)_{\text{conn}}$ is captured by the function $M(\delta_{ij})$, which following Mack we shall call the *reduced* Mellin amplitude.

The constraints (3.5) can be solved by introducing some fictitious "momentum" variables p_i living in a D-dimensional spacetime,

$$\delta_{ij} = p_i \cdot p_j \ . \tag{3.7}$$

These variables obey "momentum conservation"

$$\sum_{i=1}^{n} p_i = 0 \tag{3.8}$$

and the "on-shell" condition

$$p_i^2 = -\Delta_i \ . \tag{3.9}$$

The number of independent Lorentz invariants δ_{ij} ("Mandelstam variables") in a D-dimensional spacetime is given by

$$\begin{aligned} n < D: &\quad \frac{1}{2}n(n-3) \ , \\ n \geqslant D: &\quad n(D-1) - \frac{1}{2}D(D+1) \ . \end{aligned} \tag{3.10}$$

The counting goes as follows. The configuration space of n on-shell momenta in D dimensions is $n(D-1)$-dimensional, while the Poincaré group has dimension $\frac{1}{2}D(D+1)$. Assuming that the stability group is trivial, there will be $n(D-1) - \frac{1}{2}D(D+1)$ free parameters, giving the second line of (3.10). However for $n < D$ there is a nontrivial stability group $SO(D-n+1)$. This can be seen by using momentum conservation to make the n momenta lie in a $n-1$ dimensional hyperplane – the rotations orthogonal to the hyperplane

[2]The disconnected part is a sum of powers of x_{ij}^2 and its Mellin transform is singular.

generate the stability group $SO(D-n+1)$. Adding back the dimension of the stability group we obtain the first line of (3.10). Again we see when D is high enough, the number of independent Mandelstam variables is a D-independent number $\frac{1}{2}n(n-3)$. When $n \geq D$, the $\frac{1}{2}n(n-3)$ Mandelstam variables are subject to further relations. This is the counterpart of the statement we made about the conformal cross ratios. We conclude that the counting of independent Mandelstam variables in D dimensions coincides precisely with the counting of independent conformal cross ratios in d dimensions if we set $D = d + 1$.

The virtue of the integral representation (3.6) is to encode the consequences of the operator product expansion into simple analytic properties for $M(\delta_{ij})$. Indeed, consider the OPE

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_k c_{ij}^k \left((x_{ij}^2)^{-\frac{\Delta_i+\Delta_j-\Delta_k}{2}} \mathcal{O}_k(x_k) + \text{descendants} \right) , \qquad (3.11)$$

where for simplicity $\mathcal{O}_k$ is taken to be a scalar operator. To reproduce the leading behavior as $x_{ij}^2 \to 0$, M must have a pole at $\delta_{ij} = \frac{\Delta_i+\Delta_j-\Delta_k}{2}$, as can be seen by closing the δ_{ij} integration contour to the left of the complex plane. More generally, the location of the leading pole is controlled by the twist τ of the exchanged operator ($\tau \equiv \Delta - \ell$, the conformal dimension minus the spin). Conformal descendants contribute an infinite sequence of satellite poles, so that all in all for any primary operator $\mathcal{O}_k$ of twist τ_k that contributes to the $\mathcal{O}_i\mathcal{O}_j$ OPE the reduced Mellin amplitude $M(\delta_{ij})$ has poles at

$$\delta_{ij} = \frac{\Delta_i + \Delta_j - \tau_k - 2n}{2} , \quad n = 0, 1, 2 \ldots . \qquad (3.12)$$

Mack further defined *Mellin amplitude* $\mathcal{M}(\delta_{ij})$ by stripping off a product of Gamma functions,

$$\mathcal{M}(\delta_{ij}) \equiv \frac{M(\delta_{ij})}{\prod_{i<j} \Gamma[\delta_{ij}]} . \qquad (3.13)$$

This is a convenient definition because $\mathcal{M}$ has simpler factorization properties. In particular, for the four-point function, the s-channel OPE ($x_{12} \to 0$) implies that the Mellin amplitude $\mathcal{M}(s,t)$ has poles in s with residues that are *polynomials* of t. These *Mack polynomials* depend on the spin of the exchanged operator, in analogy with the familiar partial wave expansion of a flat-space S-matrix. (The analogy is not perfect, because each operator contributes an infinity of satellite poles, and because Mack polynomials are significantly more involved than the Gegenbauer polynomials that appear in the usual flat-space partial wave expansion.) We will see in Section 3.1.1 that Mack's definition of $\mathcal{M}$ is particularly natural for large N theories.

Finally let us comment on the integration contours in (3.6). The prescription given in [40] is that the real part of the arguments in the stripped off Gamma functions be all positive along the integration contours. To be more precise, one is instructed to integrate $\frac{1}{2}n(n-3)$ independent variables s_k along the imaginary axis, where s_k are related to δ_{ij} via

$$\delta_{ij} = \delta^0_{ij} + \sum_{k=1}^{\frac{1}{2}n(n-3)} c_{ij,k} s_k \ . \tag{3.14}$$

Here δ^0_{ij} is a special solution of the constraints (3.5) with $\Re(\delta^0_{ij}) > 0$. The coefficients $c_{ij,k}$ are any solution of

$$\begin{aligned} c_{ii,k} &= 0 \ , \\ \sum_{j=1}^{n} c_{ij,k} &= 0 \ , \end{aligned} \tag{3.15}$$

which is just the homogenous version (3.5). There are $\frac{1}{2}n(n-3)$ independent coefficients $c_{ij,k}$ for each k. We can choose to integrate over $c_{ij,k}$ with $2 \le i < j \le n$ except for $c_{23,k}$, so that the chosen $c_{ij,k}$ forms a $\frac{n(n-3)}{2} \times \frac{n(n-3)}{2}$ square matrix (the row index are the independent elements of the pair (ij) and the column index is k). We normalize this matrix to satisfy

$$|\det c_{ij,k}| = 1 \ . \tag{3.16}$$

For four-point amplitudes, which are the focus of this thesis, it is convenient to introduce "Mandelstam" variables s, t, u, and write

$$\begin{aligned} \delta_{12} &= -\frac{s}{2} + \frac{\Delta_1+\Delta_2}{2} \ , \qquad & \delta_{34} &= -\frac{s}{2} + \frac{\Delta_3+\Delta_4}{2} \ , \\ \delta_{23} &= -\frac{t}{2} + \frac{\Delta_2+\Delta_3}{2} \ , \qquad & \delta_{14} &= -\frac{t}{2} + \frac{\Delta_1+\Delta_4}{2} \ , \\ \delta_{13} &= -\frac{u}{2} + \frac{\Delta_1+\Delta_3}{2} \ , \qquad & \delta_{24} &= -\frac{u}{2} + \frac{\Delta_2+\Delta_4}{2} \ . \end{aligned} \tag{3.17}$$

With this parametrization, the constraints obeyed by δ_{ij} translate into the single constraint

$$s+t+u = \Delta_1+\Delta_2+\Delta_3+\Delta_4 \, . \tag{3.18}$$

We can take s and t as the independent integration variables, and rewrite the integration measure as

$$\int [d\delta_{ij}] = \frac{1}{4} \int_{s_0 - i\infty}^{s_0 + i\infty} ds \int_{t_0 - i\infty}^{t_0 + i\infty} dt \ . \tag{3.19}$$

In fact this simple contour prescription will need some modification. In the context of the AdS supergravity calculations, we will find it necessary to break the connected correlator into several terms and associate different contours to each term, instead of using a universal contour. There are usually poles inside the region specified by $\Re(\delta_{ij}^0) > 0$, and the answer given by the correct modified prescription differs from the naive one by the residues that are crossed in deforming the contours.

3.1.1 Large N

The Mellin formalism is ideally suited for large N CFTs. While in a general CFT the analytic structure of Mellin amplitudes is rather intricate, it becomes much simpler at large N. To appreciate this point, we recall the remarkable theorem about the spectrum of CFTs in dimension $d > 2$ proven in [54, 55]. For any two primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$ of twists τ_1 and τ_2, and for each non-negative integer k, the CFT must contain an infinite family of so-called "double-twist" operators with increasing spin ℓ and twist approaching $\tau_1 + \tau_2 + 2k$ as $\ell \to \infty$ [55, 54]. This implies that the Mellin amplitude has infinite sequences of poles accumulating at these asymptotic values of the twist, so it is *not* a meromorphic function.[3]

As emphasized by Penedones [41], a key simplification occurs in large N CFTs, where the double-twist operators are recognized as the usual double-trace operators. Thanks to large N factorization, spin ℓ conformal primaries of the schematic form $:\mathcal{O}_1 \Box^n \partial^\ell \mathcal{O}_2:$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are single-trace operators, have twist $\tau_1 + \tau_2 + 2n + O(1/N^2)$[4] for any ℓ. Recall also that the Mellin

[3]In two dimensions, there are no double-twist families, but one encounters a different pathology: the existence of infinitely many operators of the same twist, because Virasoro generators have twist zero.

[4]For definiteness, we are using the large N counting appropriate to a theory with matrix degrees of freedom, *e.g.*, a $U(N)$ gauge theory. In other kinds of large N CFTs the leading correction would have a different power – for example, $O(1/N^3)$ in the A_N six-dimensional (2, 0) theory, and $O(1/N)$ in two-dimensional symmetric product orbifolds.

amplitude is defined in terms of the *connected* part of the k-point correlator, which is of order $O(1/N^{k-2})$ for unit-normalized single-trace operators. The contribution of intermediate double-trace operators arises precisely at $O(1/N^2)$, so that to this order we can use their uncorrected dimensions. Remarkably, the poles corresponding to the exchanged double-trace operators are precisely captured by the product of Gamma functions $\prod_{i<j}\Gamma(\delta_{ij})$ that Mack stripped off to define the Mellin amplitude $\mathcal{M}$. All in all, we conclude that the $O(1/N^{k-2})$ Mellin amplitude $\mathcal{M}$ *is* a meromorphic function, whose poles are controlled by just the exchanged *single-trace* operators.

Let us analyze in some detail the case of the four-point function. For four scalar operators $\mathcal{O}_i$ of dimensions Δ_i, conformal covariance implies

$$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \frac{(x_{24}^2)^{\frac{\Delta_1-\Delta_2}{2}}(x_{14}^2)^{\frac{\Delta_3-\Delta_4}{2}}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}(x_{14}^2)^{\frac{\Delta_1-\Delta_2}{2}}(x_{13}^2)^{\frac{\Delta_3-\Delta_4}{2}}}\mathcal{G}(U,V)\ , \tag{3.20}$$

where U and V are the usual conformal cross-ratios[5]

$$U=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,\quad V=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\,. \tag{3.21}$$

Taking the operators $\mathcal{O}_i$ to be unit-normalized single-trace operators, and separating out the disconnected and connected terms,[6]

$$\mathcal{G}=\mathcal{G}_{\rm disc}+\mathcal{G}_{\rm conn}\ , \tag{3.22}$$

we have the following familiar large N counting:

$$\mathcal{G}_{\rm disc}=O(1)\,,\quad \mathcal{G}_{\rm conn}=\frac{1}{N^2}\mathcal{G}^{(1)}+\frac{1}{N^4}\mathcal{G}^{(2)}+\dots \tag{3.23}$$

We also recall that the Mellin amplitude $\mathcal{M}$ is defined for the connected part of the correlator by the integral transform

$$\begin{aligned}\mathcal{G}_{\rm conn} = \int_{-i\infty}^{i\infty}&\frac{ds}{2}\frac{dt}{2}U^{\frac{s}{2}}V^{\frac{t}{2}-\frac{\Delta_2+\Delta_3}{2}}\mathcal{M}(s,t)\Gamma[\frac{\Delta_1+\Delta_2-s}{2}]\Gamma[\frac{\Delta_3+\Delta_4-s}{2}]\\ &\times\Gamma[\frac{\Delta_1+\Delta_4-t}{2}]\Gamma[\frac{\Delta_2+\Delta_3-t}{2}]\Gamma[\frac{\Delta_1+\Delta_3-u}{2}]\Gamma[\frac{\Delta_2+\Delta_4-u}{2}]\ ,\end{aligned} \tag{3.24}$$

[5]We use capital letters because the symbol u is already taken to denote the Mandelstam invariant, (3.17).

[6]The disconnected term $\mathcal{G}_{\rm disc}$ will of course vanish unless the four operators are pairwise identical.

with $s+t+u = \Delta_1+\Delta_2+\Delta_3+\Delta_4$.

Let us first assume that the dimensions Δ_i are generic. In the s-channel OPE, we expect contributions to $\mathcal{G}_{\rm conn}$ from the tower of double-trace operators of the form[7] $:\mathcal{O}_1\Box^n\partial^\ell\mathcal{O}_2:$, with twists $\tau = \Delta_1+\Delta_2+2n+O(1/N^2)$, and from the tower $:\mathcal{O}_3\Box^n\partial^\ell\mathcal{O}_4:$, which have twists $\tau = \Delta_1+\Delta_2+2n+O(1/N^2)$. The OPE coefficients scale as

$$\begin{aligned}
&\langle \mathcal{O}_1\mathcal{O}_2 : \mathcal{O}_1\Box^n\partial^\ell\mathcal{O}_2 :\rangle = O(1), \qquad \langle \mathcal{O}_3\mathcal{O}_4 : \mathcal{O}_1\Box^n\partial^\ell\mathcal{O}_2 :\rangle = O(N^{-2}), \\
&\langle \mathcal{O}_3\mathcal{O}_4 : \mathcal{O}_3\Box^n\partial^\ell\mathcal{O}_4 :\rangle = O(1), \qquad \langle \mathcal{O}_1\mathcal{O}_2 : \mathcal{O}_3\Box^n\partial^\ell\mathcal{O}_4 :\rangle = O(N^{-2}),
\end{aligned} \tag{3.25}$$

so that to leading $\mathcal{O}(1/N^2)$ order, we can neglect the $1/N^2$ corrections to the conformal dimensions of the double-trace operators. All in all, we expect that these towers of double-trace operators contribute poles in s at

$$\begin{aligned}
s &= \Delta_1+\Delta_2+2m_{12}\,, \qquad && m_{12}\in\mathbb{Z}_{\geqslant 0}\,, \\
s &= \Delta_3+\Delta_4+2m_{34}\,, \qquad && m_{34}\in\mathbb{Z}_{\geqslant 0}.\,.
\end{aligned} \tag{3.26}$$

These are precisely the locations of the poles of the first two Gamma functions in (3.24). In complete analogy, the poles in t and u in the other Gamma functions account for the contributions of the double-trace operators exchanged in the t and u channels.

If $\Delta_1+\Delta_2-(\Delta_3+\Delta_4) = 0$ mod 2, the two sequences of poles in (3.26) (partially) overlap, giving rise to a sequence of *double* poles at

$$s = \max\{\Delta_1+\Delta_2, \Delta_3+\Delta_4\}+2n\,, \qquad n\in\mathbb{Z}_{\geqslant 0}\,. \tag{3.27}$$

A double pole at $s=s_0$ gives a contribution to $\mathcal{G}_{\rm conn}(U,V)$ of the from $U^{s_0/2}\log U$. This has a natural interpretation in terms of the $O(1/N^2)$ anomalous dimensions of the exchanged double-trace operators. Indeed, a little thinking shows that in this case both OPE coefficients in the s-channel conformal block expansion are of order one (in contrast with the generic case (3.26)), so that the $O(1/N^2)$ correction to the dilation operator gives a leading contribution to the connected four-point function.

Let's see this more explicitly. Let's take for definiteness $\Delta_1+\Delta_2 \leqslant \Delta_3+\Delta_4$, so that $\Delta_3+\Delta_4 = \Delta_1+\Delta_2+2k$ for some non-negative integer k. Then the double-trace operators of the schematic form

$$:\mathcal{O}_1\Box^{n+k}\partial^\ell\mathcal{O}_2: \quad \text{and} \quad :\mathcal{O}_3\Box^n\partial^\ell\mathcal{O}_4: \tag{3.28}$$

[7] In fact for fixed n and ℓ, there are in general multiple conformal primaries of this schematic form, which differ in the way the derivatives are distributed.

have the same conformal dimension to leading large N order, as well as the same Lorentz quantum numbers. They are then expected to mix under the action of the $O(1/N^2)$ dilation operator. It is important to realize that the mixing matrix that relates the basis (3.28) to the double-trace dilation eigenstates $\mathcal{O}_\alpha^{\rm DT}$ is of order one. The OPE coefficients $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_\alpha^{\rm DT}\rangle = c_{12\alpha}$ and $\langle \mathcal{O}_3\mathcal{O}_4\mathcal{O}_\alpha^{\rm DT}\rangle = c_{34\alpha}$ are then both $O(1)$, as claimed. The twist $\tau_\alpha = \Delta_\alpha - \ell$ has a large N expansion of the form $\tau_\alpha = \Delta_3+\Delta_4+2n+\gamma_\alpha^{(1)}/N^2+O(1/N^4)$. All is all, we find a contribution to $\mathcal{G}_{\rm conn}$ of the form

$$\frac{c_{12\alpha}c_{34\alpha}\,\gamma_\alpha^{(1)}}{N^2}\, U^{\frac{\Delta_3+\Delta_4}{2}+n}\log U\,. \tag{3.29}$$

In Mellin space, this corresponds to a double-pole at $s = \Delta_3 + \Delta_4 + 2n$, just as needed. In summary, the explicit Gamma functions that appear in Mack's definition provide precisely the analytic structure expected in a large N CFT, if we take the $O(1/N^2)$ Mellin amplitude $\mathcal{M}$ to have poles associated with just the exchanged *single-trace* operators. The upshot is that to leading $O(1/N^2)$ order, fixing the single-trace contributions to the OPE is sufficient determine the double-trace contributions as well.[8]

By following a similar reasoning, we will now argue that compatibility with the large N OPE imposes some further constraints on the analytic structure of $\mathcal{M}$. We have seen that to leading $O(1/N^2)$ order the Mellin amplitude $\mathcal{M}(s,t,u)$ is a meromorphic function with only simple poles associated to the exchanged single-trace operators. In the generic case, a single-trace operator $\mathcal{O}^{\rm ST}$ of twist τ contributing to the s-channel OPE is responsible for an infinite sequence of simple poles at $s=\tau+2n$, $n\in\mathbb{Z}_{\geqslant 0}$ (and similarly for the other channels). But this rule needs to be modified if this sequence of "single-trace poles" overlaps with the "double-trace poles" from the explicit Gamma functions in (3.24). This happens if $\tau = \Delta_1+\Delta_2$ mod 2, or if $\tau = \Delta_3+\Delta_4$ mod 2. (We assume for now that $\Delta_1+\Delta_2 \neq \Delta_3+\Delta_4$ mod 2, so that only one of the two options is realized.) In the first case, the infinite sequence of poles in $\mathcal{M}$ must truncate to the set $\{\tau, \tau+2, \ldots, \tau+\Delta_1+\Delta_2-2\}$, and in the second case to the set $\{\tau, \tau+2, \ldots, \tau+\Delta_3+\Delta_4-2\}$[9]. This

[8]This is particularly apparent in Mellin space but can also be argued by more abstract CFT reasoning [56, 57, 50, 58].

[9]Note that the first set empty if $\Delta_1+\Delta_2 < \tau$ (again we are assuming $\Delta_1+\Delta_2 = \tau$ mod 2) and the second is empty if $\Delta_3+\Delta_4<\tau$ (with $\Delta_3+\Delta_4=\tau$ mod 2). In these cases, $\mathcal{O}^{\rm ST}$ does not contribute *any* poles to $\mathcal{M}$.

truncation must happen because double poles in s, translating to $\sim \log U$ terms in $\mathcal{G}_{\rm conn}$, are incompatible with the large N counting. Indeed, the OPE coefficients already provide an $O(1/N^2)$ suppression, so that we should use the $O(1)$ dilation operator, and no logarithmic terms can arise in $\mathcal{G}_{\rm conn}$ to leading $O(1/N^2)$ order.[10]

3.1.2 Mellin Amplitudes for Witten Diagrams

The effectiveness of Mellin formalism is best illustrated by its application to the calculation of Witten diagrams. Conceptually, Mellin space makes transparent the analogy of holographic correlators and S-matrix amplitudes. Practically, Mellin space expressions for Witten diagrams are much simpler than their position space counterparts. For starters, the Mellin amplitude of a four-point contact diagram, which is the building blocks of AdS four-point correlators as we reviewed in Section 2.1, is just a constant,

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4} = \int [d\delta_{ij}] \left(\frac{\pi^{d/2}\Gamma[\frac{\sum \Delta_i}{2} - d/2]}{\prod \Gamma[\Delta_i]} \right) \times \prod_{i<j} \Gamma[\delta_{ij}](x_{ij}^2)^{-\delta_{ij}} \ . \qquad (3.30)$$

As was shown in [41], this generalizes to n-point contact diagram with a non-derivative vertex: their Mellin amplitude is again a constant. Contact diagrams with derivative vertices are also easily evaluated. It will be important in the following that the Mellin amplitude for a contact diagram arising from a vertex with $2n$ derivatives is an order n polynomial in the Mandelstam variables δ_{ij}.

Exchange diagrams are also much simpler in Mellin space. The s-channel exchange Witten diagram with an exchanged field of conformal dimension Δ and spin J has a Mellin amplitude with the following simple analytic structure [44],

$$\mathcal{M}(s,t) = \sum_{m=0}^{\infty} \frac{Q_{J,m}(t)}{s-\tau-2m} + P_{J-1}(s,t)\,, \qquad (3.31)$$

[10] In the even more fine-tuned case $\tau = \Delta_1+\Delta_2 = \Delta_3+\Delta_4$ mod 2, clearly the poles in s in the $O(1/N^2)$ Mellin amplitude $\mathcal{M}$ must truncate to the set $\{\tau, \tau+2, \ldots, \tau+\min\{\Delta_1+\Delta_2, \Delta_3+\Delta_4\}-2\}$. The double poles at $\{\min\{\Delta_1+\Delta_2,\Delta_3+\Delta_4\}, \min\{\Delta_1+\Delta_2,\Delta_3+\Delta_4\}+2\ldots, \max\{\Delta_1+\Delta_2,\Delta_3+\Delta_4\}-2\}$ can be ruled out by the same reasoning, while the *triple* poles at $s=\max\{\Delta_1+\Delta_2,\Delta_3+\Delta_4\}+2n$ would give rise to $\sim (\log U)^2$ terms, which absolutely cannot appear to $O(1/N^2)$.

where $\tau = \Delta - J$ is the twist. Here $Q_{J,m}(t)$ are polynomials in t of degree J and $P_{J-1}(s,t)$ polynomials in s and t of degree $J-1$. These polynomials depend on the dimensions $\Delta_{1,2,3,4}$, Δ, as well as the spin J. The detailed expressions for these polynomials are quite complicated but will not be needed for our analysis. The $m = 0$ pole at $s = \tau$ is called the leading pole, corresponding to the primary operator that is dual to the exchanged field, while the $m > 0$ poles are called satellite poles, and they are associated with conformal descendants.

It has been observed (see, *e.g.*, [41]) that the infinite series of poles in (3.31) truncates to a finite sum if $\tau = \Delta_1 + \Delta_2 \mod 2$ or if $\tau = \Delta_3 + \Delta_4 \mod 2$. One finds that the upper limit of the sum $m_{\max}$ is given by $\tau - \Delta_1 - \Delta_2 = 2(m_{\max} + 1)$ in the first case and by $\tau - \Delta_3 - \Delta_4 = 2(m_{\max} + 1)$ in the second case. This is the Mellin space version of the phenomenon described in Section 2.1: an exchange Witten diagram with these special values of quantum numbers can be written as a *finite* sum of contact Witten diagrams. As we have explained in the previous subsection, this remarkable simplification is dictated by compatibility with the large N OPE in the dual CFT.

3.1.3 Asymptotics and the Flat Space Limit

In the next section we will determine the supergravity four-point Mellin amplitude using general consistency principles. A crucial constraint will be provided by the asymptotic behavior of $\mathcal{M}(s,t)$ when s and t are simultaneously scaled to infinity. On general grounds, one can argue [41] that in this limit the Mellin amplitude should reduce to the flat-space bulk S-matrix (in $\mathbb{R}^{d,1}$).

A precise prescription for relating the massless[11] flat-space scattering amplitude $\mathcal{T}(K_i)$ to the asymptotic form of the holographic Mellin amplitude was given in [41] and justified in [45],

$$\mathcal{M}(\delta_{ij}) \approx \frac{R^{n(1-d)/2+d+1}}{\Gamma(\frac{1}{2}\sum_i \Delta_i - \frac{d}{2})} \int_0^\infty d\beta \beta^{\frac{1}{2}\sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta} \mathcal{T}\left(S_{ij} = \frac{2\beta}{R^2} s_{ij}\right) . \tag{3.32}$$

where $S_{ij} = -(K_i + K_j)^2$ are the Mandelstam invariants of the flat-space scattering process. We have a precise opinion for asymptotic behavior of the

[11]For massive external particles, see the discussion in [59].

flat-space four-point amplitude $\mathcal{T}(S,T)$ – it can grow at most linearly for large S and T. Indeed, a spin ℓ exchange diagrams grows with power $\ell - 1$, and the highest spin state is of course the graviton with $\ell = 2$. Similarly, contact interactions with $2n$ derivatives give a power n growth. IIB supergravity (in ten-dimensional flat space) and eleven dimensional supergravity contain contact interactions with at most two derivatives. From (3.32) we then deduce

$$\mathcal{M}(\beta s, \beta t) \sim O(\beta) \quad \text{for } \beta \to \infty\,. \tag{3.33}$$

It is of course crucial to this argument that we are calculating within the standard two-derivative supergravity theory. Stringy α'-corrections and M-theory correction would introduce higher derivative terms and invalidate this conclusion.[12]

Curiously, the asymptotic behavior (3.33) is not immediately obvious if one for example computes holographic correlators in $AdS_5 \times S^5$ by the standard diagrammatic approach. Exchange Witten diagrams have the expected behavior, with growth at most linear from spin two exchanges, see (3.31).[13] However, the AdS_5 effective action [32] obtained by Kaluza-Klein reduction of IIB supergravity on S^5 contains quartic vertices with four derivatives (or fewer). The four-derivative vertices are in danger of producing an $O(\beta^2)$ growth, which would ruin the expected flat space asymptotics. On this basis, we made the assumption in [1] that the total contribution of the four-derivative vertices to a holographic correlator must also grow at most linearly for large β. Indeed, this was experimentally the case in all the explicit supergravity calculations performed at the time. Fortunately, the conjectured cancellation of the $O(\beta^2)$ terms has been recently proved in full generality [35].

[12] In a perturbative α'-expansion, we expect increasing polynomial growth, but for finite α' the behavior should be very soft, as in string theory.

[13] The AdS_5 effective theory contains an infinite tower of spin two massive states that arise from the Kaluza-Klein reduction of the ten-dimensional graviton, and of course *no* states of spin higher than two.

3.2 Digression: Mellin Formalism for CFTs with a Conformal Interface

In this section we make a digression to discuss the extension of the Mellin representation formalism to conformal field theories whose conformal symmetry is partially broken by a boundary or a co-dimension one defect (an interface) [3].[14] There are compelling motivations for developing the Mellin technology for conformal theories with interfaces and boundaries. First, these theories are very interesting in their own right. They have important physical applications in statistical mechanics and condensed matter physics (see, *e.g.*, [61]), formal field theory (see, *e.g.*, [62] for supersymmetric examples) worldsheet string theory, (where D-branes are defined by boundaries on the string world sheet) and holography [63, 64], to give only an unsystematic sampling of a large literature. Second, boundary and interface conformal field theories are a useful theoretical arena to develop the bootstrap program [65, 66, 67, 68, 69, 70, 71, 72], especially if one's goal is to gain analytic insight. Indeed, the simplest non-trivial ICFT correlator is the two-point function with two "bulk" insertions (*i.e.*, two operators inserted at $x_\perp \neq 0$). Being a function of a single cross ratio, it is more tractable than the four-point function in an ordinary CFT, which has two cross ratios.

The organization of this section is as follows. In Section 3.2.1 we review the embedding formalism of CFTs with a conformal interface which makes the subsequent discussion easier. In Section 3.2.2 we set up the Mellin representation formalism. Finally in Section 3.2.3 we apply this formalism to a simple holographic setup where we perform a systematic study of the Witten diagrams.

3.2.1 Conformal Covariance in Embedding Space

To facilitate the discussion in this section, it is useful to first introduce the embedding formalism under which the action of the conformal group is linearized. We start by deriving the general form of the correlation function of n bulk scalar operators and m interface scalar operators. (For definiteness, we will use the language appropriate to the interface case, but all formulae will be valid for the boundary case with the obvious modifications). We will use the standard Euclidean coordinates $x^\mu = (x^1, \ldots, x^{d-1}, x_\perp)$ and place the

[14] Further generalization to defects with generic co-dimensions appeared recently in [60].

interface at $x_\perp = 0$. The coordinates parallel to the interface will be denoted as $\vec{x}$. As is familiar, a convenient way to make the conformal symmetry manifest is to to lift this space to an "embedding space" of dimension $d+2$ and signature $(-,+,+,\ldots)$. In the embedding space, points are labelled by lightcone coordinates which we denote as $P^A = (P^+, P^-, P^1, \ldots, P^d)$. The physical space has only d coordinates and is restricted to be on a projective null cone in the embedding space,

$$P^A P_A = 0 \qquad \text{with} \qquad P^A \sim \lambda P^A \ . \tag{3.34}$$

The physical space coordinates x are related to the embedding space by the map

$$x^\mu = \frac{P^\mu}{P^+} \ . \tag{3.35}$$

Using the scaling freedom we can fix P^+ to be 1, so that

$$P^A = (1, \vec{x}^2 + x_\perp^2, \vec{x}, x_\perp) \ . \tag{3.36}$$

The conformal group $SO(d+1,1)$ which acts non-linearly on x^μ is now realized linearly as the Lorentz group on the embedding coordinates P^A. Conformal invariants in the physical space can be conveniently constructed from the embedding space as $SO(d+1,1)$ invariants.

In the presence of the interface, the conformal group is broken down to the subgroup $SO(d,1)$. In the embedding space language, this can be conveniently described by introducing a fixed vector B^A,

$$B^A = (0, 0, \vec{0}, 1) \ . \tag{3.37}$$

Points on the boundary uplift to vectors $\widehat{P}_A = (1, \vec{x}^2, \vec{x}, 0)$ that are transverse to B^A, $\widehat{P}_A B^A = 0$. The residual conformal transformations $SO(d,1)$ correspond to the linear transformations of P^A that keep B^A fixed.

In this section we shall focus on scalar operators. Scalar operators are assumed to transform homogeneously under rescaling in the embedding space,

$$O_\Delta(\lambda P) = \lambda^{-\Delta} O_\Delta(P) \ , \qquad \widehat{O}_{\widehat{\Delta}}(\lambda \widehat{P}) = \lambda^{-\widehat{\Delta}} \widehat{O}_{\widehat{\Delta}}(\widehat{P}) \ . \tag{3.38}$$

Here O_Δ is a bulk operator and $\widehat{O}_{\widehat{\Delta}}$ an interface operator, with Δ and $\widehat{\Delta}$ their respective conformal dimensions. (Hatted quantities will always be interface quantities).

The correlator of n bulk and m interface operators,

$$\mathcal{C}_{n,m} \equiv \langle O_{\Delta_1}(P_1) \dots O_{\Delta_n}(P_n) \widehat{O}_{\widehat{\Delta}_1}(\widehat{P}_1) \dots \widehat{O}_{\widehat{\Delta}_m}(\widehat{P}_m) \rangle \ , \tag{3.39}$$

should be invariant under the residual $SO(d,1)$ symmetry and have the correct scaling weights when we rescale the embedding coordinate of each operator. There are only a handful of $SO(d,1)$ invariant structures,

$$\begin{aligned} -2P_i \cdot P_j &= (x_i - x_j)^2 \equiv (\vec{x}_i - \vec{x}_j)^2 + (x_{\perp i} - x_{\perp j})^2 & (3.40) \\ -2P_i \cdot \widehat{P}_I &= (\vec{x}_i - \vec{x}_J)^2 + (x_{\perp i})^2 & (3.41) \\ -2\widehat{P}_I \cdot \widehat{P}_J &= (\vec{x}_I - \vec{x}_J)^2 & (3.42) \\ P_i \cdot B &= x_{\perp i} \, , & (3.43) \end{aligned}$$

where $i = 1, \dots n$ and $I = 1, \dots m$. The most general form of the scalar correlator is

$$\mathcal{C}_{n,m} = \left(\prod_{i<j} (-2P_i \cdot P_j)^{-\delta^0_{ij}} \prod_{i,I} (-2P_i \cdot \widehat{P}_I)^{-\gamma^0_{iI}} \prod_{I<J} (-2\widehat{P}_I \cdot \widehat{P}_J)^{-\beta^0_{IJ}} , \prod_i (P_i \cdot B)^{-\alpha^0_i} \right) f(\xi_r) \tag{3.44}$$

where the exponents must obey

$$\begin{aligned} &\sum_j \delta^0_{ij} + \sum_I \gamma^0_{iI} + \alpha^0_i = \Delta_i \ , \\ &\sum_i \gamma^0_{iI} + \sum_J \beta^0_{IJ} = \widehat{\Delta}_I \end{aligned} \tag{3.45}$$

in order to give the correct scaling weights, while f is an arbitrary function that depends on the cross ratios ξ_r, which are ratios of the invariants (3.40) with zero scaling weights.

Let us also recall that anti de Sitter space admits a simple description using the embedding coordinates. Euclidean AdS_{d+1} is just the hyperboloid defined by the equation

$$Z^2 = -R^2, \qquad Z^0 > 0, \qquad Z \in \mathbb{R}^{1,d+1} \ . \tag{3.46}$$

We will usually set $R = 1$. The Poincaré coordinates of AdS_{d+1} are related to the embedding coordinates as

$$Z^A = \frac{1}{z_0}(1, z_0^2 + \vec{z}^2 + z_\perp^2, \vec{z}, z_\perp) \ . \tag{3.47}$$

3.2.2 Mellin Formalism for Interface CFTs

After these preliminaries, we are ready to define the Mellin representation for interface CFTs. Recalling that a scalar correlator $\mathcal{C}_{n,m}$ with n bulk and m interface insertions takes the general form (3.44), it is natural to write it in terms of the following integral transform,

$$\mathcal{C}_{n,m} = \int \prod_{i<j} d\delta_{ij}(-2P_i \cdot P_j)^{-\delta_{ij}} \prod_{i,I} d\gamma_{iI}(-2P_i \cdot \widehat{P}_I)^{-\gamma_{iI}} \prod_{I<J} d\beta_{IJ}(-2\widehat{P}_I \cdot \widehat{P}_J)^{-\beta_{IJ}} \\ \times \prod_i d\alpha_i (P_i \cdot B)^{-\alpha_i} M(\delta_{ij}, \gamma_{iI}, \beta_{IJ}, \alpha_i) \ . \tag{3.48}$$

The variables $\delta_{ij},\ \gamma_{iI},\ \beta_{IJ},\ \alpha_i$ are constrained to obey

$$\begin{aligned} \sum_j \delta_{ij} + \sum_I \gamma_{iI} + \alpha_i &= \Delta_i \ , \\ \sum_i \gamma_{iI} + \sum_J \beta_{IJ} &= \widehat{\Delta}_I \ . \end{aligned} \tag{3.49}$$

A simple counting tells that there are

$$\frac{n(n-1)}{2} + \frac{m(m-1)}{2} + nm - m \tag{3.50}$$

independent such variables, in one-to-one correspondence with the independent conformal cross ratios so long as the spacetime dimension is high enough, namely for $d > n + m$.

By a natural generalization of the case with no interface, the constraints (3.49) can be solved in terms of some fictitious momenta. We assign to each bulk operator a $(d+1)$-dimensional momentum p_i, to each interface operator a d-dimensional momentum $\widehat{p}_I$ and to the interface itself a $(d+1)$-dimensional momentum $\mathcal{P}$. The momenta need to be conserved and on-shell,

$$\sum_i p_i + \sum_I \widehat{p}_I + \mathcal{P} = 0\,, \quad p_i^2 = -\Delta_i\ , \quad \widehat{p}_I^2 = -\widehat{\Delta}_I\,. \tag{3.51}$$

Moreover, the momenta of the interface operators must be orthogonal to $\mathcal{P}$

$$\widehat{p}_I \cdot \mathcal{P} = 0\ . \tag{3.52}$$

Then we can write

$$\delta_{ij} = p_i \cdot p_j \ , \quad \gamma_{iI} = p_i \cdot \widehat{p}_I \ , \quad \beta_{IJ} = \widehat{p}_I \cdot \widehat{p}_J \ , \quad \alpha_i = p_i \cdot \mathcal{P} \ . \tag{3.53}$$

From (3.51) and (3.52), we can replace $\mathcal{P}^2$ by $-\sum_i \mathcal{P} \cdot p_i$ and $\mathcal{P} \cdot p_i$ by $-\sum_j p_i \cdot p_j - \sum_I p_i \cdot \widehat{p}_I$. For the remaining bilinears there are still m equations relating $p_i \cdot \widehat{p}_I$ to $\widehat{p}_I \cdot \widehat{p}_J$: $\sum_i p_i \cdot \widehat{p}_J + \sum_I \widehat{p}_I \cdot \widehat{p}_J = 0$. So the number of independent momentum bilinears is

$$\frac{n(n-1)}{2} + \frac{m(m-1)}{2} + nm - m \quad \text{if } d > n + m \, , \tag{3.54}$$

in agreement with the number of independent conformal cross ratios. This is the appropriate counting for $d > n+m$. For $d \leq n+m$, *both* the counting of independent Mandelstam invariants (3.53) and the counting of conformal cross-ratios for a configuration of n bulk and m interface operators give instead

$$nd + m(d-1) - \frac{1}{2}d(d+1) \quad \text{if } d \leq n+m \, . \tag{3.55}$$

Clearly, the parametrization (3.53) corresponds to the kinematic setup of a scattering process off a fixed target, with n particles having arbitrary momenta p_i, m particles having momenta $\hat{p}_I$ parallel to the target and $\mathcal{P}$ the momentum transfer in the direction perpendicular of the infinitely heavy target.

Following Mack's terminology, we call M the *reduced* Mellin amplitude. In our case, we wish to define the Mellin amplitude $\mathcal{M}$ by

$$\mathcal{M} = \frac{M(\delta_{ij}, \gamma_{iJ}, \beta_{IJ}, \alpha_i)}{\prod_{i<j} \Gamma(\delta_{ij}) \prod_{i,J} \Gamma(\gamma_{iJ}) \prod_{I<J} \Gamma(\beta_{IJ}) \prod_i \Gamma(\alpha_i)} \cdot \frac{\Gamma(-\mathcal{P}^2)}{\Gamma(-\frac{\mathcal{P}^2}{2})} \, . \tag{3.56}$$

The Gamma functions in the denominator of the first fraction are the counterpart of the Gamma function (3.13), accounting for the expected contributions in a holographic interface theory. To wit, the poles in $\Gamma(\delta_{ij})$ correspond to double-trace bulk operators of the form $\mathcal{O}_i \Box^n \mathcal{O}_j$, the poles in $\Gamma(\gamma_{iJ})$ to double-trace bulk-interface operators of the form $\mathcal{O}_i \Box^n \hat{\mathcal{O}}_J$, and the poles in $\Gamma(\beta_{IJ})$ to double-trace interface-interface operators of the form $\hat{\mathcal{O}}_I \Box^n \hat{\mathcal{O}}_J$. The poles in $\Gamma(\alpha_i)$ correspond to interface operators of the form $\partial_\perp^n \mathcal{O}_i(\vec{x}, x_\perp = 0)$, *i.e.*, to the restriction to the interface of a bulk operator and its normal derivatives – these operators are indeed present in the simple holographic setup for ICFT that we consider below.[15] The factor of $\Gamma(-\mathcal{P}^2)/\Gamma(-\frac{\mathcal{P}^2}{2})$ has a different justification. We have introduced it to ensure that the Mellin amplitude of a contact Witten diagram is just a constant, as we will show in Section 3.2.3 below. Note that if there are only interface operators ($n=0$), the constraints imply $\mathcal{P}\equiv 0$. The additional factor $\Gamma(-\mathcal{P}^2)/\Gamma(-\frac{\mathcal{P}^2}{2})$ becomes simply 2 and our definition of $\mathcal{M}$ reduces to Mack's, up to an overall normalization.

Let us specialize the formalism to the important case of two bulk insertions ($n=2$) and no interface insertion ($m=0$). We have

$$\begin{aligned}\langle O_{\Delta_1} O_{\Delta_2}\rangle = &\int d[\alpha,\delta](-2P_1\cdot P_2)^{-\delta_{12}}(P_1\cdot B)^{-\alpha_1}(P_2\cdot B)^{-\alpha_2}\\ &\times\Gamma(\delta_{12})\Gamma(\alpha_1)\Gamma(\alpha_2)\frac{\Gamma(-\frac{\mathcal{P}^2}{2})}{\Gamma(-\mathcal{P}^2)}\mathcal{M}(\alpha,\delta)\ .\end{aligned} \tag{3.57}$$

The variables $\delta_{12},\ \alpha_1,\ \alpha_2$ must obey

$$\delta_{12}+\alpha_1=\Delta_1\ ,\qquad \delta_{12}+\alpha_2=\Delta_2\,. \tag{3.58}$$

The constraints can be solved using the parameterization

$$\delta_{12}=-p_1\cdot p_2\ ,\quad \alpha_1=-p_1\cdot\mathcal{P}\ ,\quad \alpha_2=-p_2\cdot\mathcal{P}\,, \tag{3.59}$$

with the constraints

$$p_1+p_2+\mathcal{P}=0\ ,\quad p_1^2=-\Delta_1,\quad p_2^2=-\Delta_2\ . \tag{3.60}$$

These constraints leave only one independent variable that is bilinear in the momenta, namely $\mathcal{P}^2$,

$$\begin{aligned}p_1\cdot p_2 &= \frac{\mathcal{P}^2+\Delta_1+\Delta_2}{2}\ ,\\ p_1\cdot\mathcal{P} &= \frac{-\mathcal{P}^2+\Delta_1-\Delta_2}{2}\ ,\\ p_2\cdot\mathcal{P} &= \frac{-\mathcal{P}^2-\Delta_1+\Delta_2}{2}\ .\end{aligned} \tag{3.61}$$

[15] In a holographic BCFT setup, one would need to impose boundary conditions that would remove some of these operators, *i.e.*, Dirichlet boundary conditions would remove $\mathcal{O}_i(\vec{x},x_\perp=0)$ while Neumann boundary conditions would remove $\partial_\perp\mathcal{O}_i(\vec{x},x_\perp=0)$.

It turns out to be convenient to make a change of variable

$$\mathcal{P}^2 = 2\tau - \Delta_1 - \Delta_2\,. \tag{3.62}$$

The Mellin representation becomes

$$\langle O_{\Delta_1} O_{\Delta_2}\rangle = \frac{1}{(2x_{1,\perp})^{\Delta_1}(2x_{2,\perp})^{\Delta_2}} \int_{-i\infty}^{i\infty} d\tau \left(\frac{\eta}{4}\right)^{-\tau} \frac{\Gamma(\tau)\Gamma(\Delta_1-\tau)\Gamma(\Delta_2-\tau)}{\Gamma(\frac{1+\Delta_1+\Delta_2}{2}-\tau)}\mathcal{M}(\tau)\,, \tag{3.63}$$

where η is the standard conformal cross ratio,

$$\eta = \frac{(x_1-x_2)^2}{x_{1,\perp}x_{2,\perp}} = \frac{(\vec{x}_1-\vec{x}_2)^2+(x_{1,\perp}-x_{2,\perp})^2}{x_{1,\perp}x_{2,\perp}}\,. \tag{3.64}$$

3.2.3 Application to Witten Diagrams in the Probe Brane Setup

In this subsection, we consider the simplest holographic framwork which is the simplest version of the Karch-Randall setup [63, 73]. The dual geometry is taken to be AdS_{d+1} with a preferred AdS_d subspace. In string theory, this geometry can be obtained by taking the near horizon limit of a stack of N "color" D-branes, intersecting a single "flavor" brane along the interface. At large N, the backreaction of the flavor brane can be ignored. Schematically, the effective action is taken to be

$$S = \int_{AdS_{d+1}} \mathcal{L}_{\text{bulk}}[\Phi_i] + \int_{AdS_d} \left(\mathcal{L}_{\text{interface}}[\phi_I] + \mathcal{L}_{\text{bulk/interface}}[\Phi_i,\phi_I]\right)\,. \tag{3.65}$$

where Φ_i denotes the fields that live in the full space AdS_{d+1}, while ϕ_I denotes the additional fields living on the AdS_d brane. The holographic dictionary associates to Φ_i the local operators $\mathcal{O}_i$ of the bulk[16] CFT$_d$, and to ϕ_I the operators living on the $(d-1)$-dimensional interface at $x_\perp = 0$. We perform a systematic study of the Witten diagrams in this geometry. Some special cases of the Witten diagrams have been previously studied in [74].

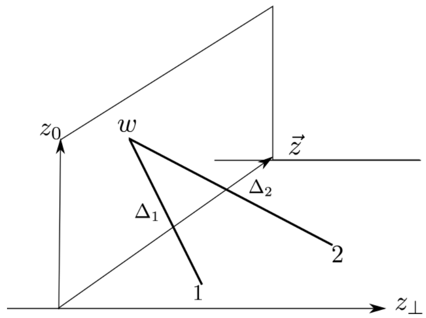

Figure 3.1: A contact Witten diagram with $n = 2$ points in the bulk and $m = 0$ points on the interface.

Contact Witten Diagrams

We consider contact Witten diagram with n operators in the bulk and m operators on the interface. In Figure 3.1, we illustrated the case where we have 2 points in the bulk and no point on the interface. In the following calculation, we only assume $n \geq 1$ since otherwise the calculation reduces to the known case where an interface is absent. Such a contact Witten diagram was denoted as $W_{n,m}$ and it is the following integral

$$\begin{aligned} W_{n,m} &\equiv \langle O_{\Delta_1}(x_1)\ldots O_{\Delta_n}(x_n)\widehat{O}_{\widehat{\Delta}_1}(y_1)\ldots\widehat{O}_{\widehat{\Delta}_m}(y_m)\rangle_{\text{contact}} \\ &= \int\frac{dw_0 d\vec{w}}{w_0^d}\prod_i\left(\frac{w_0}{w_0^2+x_{\perp,i}^2+(\vec{w}-\vec{x}_i)^2}\right)^{\Delta_i}\prod_I\left(\frac{w_0}{w_0^2+(\vec{w}-\vec{y}_I)^2}\right)^{\widehat{\Delta}_I}. \end{aligned} \tag{3.66}$$

We use Schwinger's trick to bring the denominators into the exponent and perform the integrals of dw_0 and $d\vec{w}$, which leads to

$$\begin{aligned} W_{n,m} = &\frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{\sum_i\Delta_i+\sum_I\widehat{\Delta}_I-d+1}{2})}{2\prod_i\Gamma(\Delta_i)\prod_I\Gamma(\widehat{\Delta}_I)}\int_0^\infty\prod_i\frac{dt_i}{t_i}t_i^{\Delta_i}\frac{ds_I}{s_I}s_I^{\widehat{\Delta}_I}(\sum_i t_i+\sum_I s_I)^{-\frac{\sum_i\Delta_i+\sum_I\widehat{\Delta}_I}{2}} \\ &\times\exp\left[-\frac{\sum\limits_{i<j}t_it_j(-2P_i\cdot P_j)+(\sum\limits_i t_iP_i\cdot B)^2+\sum\limits_{I<J}s_Is_J(-2\widehat{P}_I\cdot\widehat{P}_J)+\sum\limits_{i,I}t_is_I(-2P_i\cdot\widehat{P}_I)}{\sum\limits_i t_i+\sum\limits_I s_I}\right]. \end{aligned} \tag{3.67}$$

[16]Holographic boundary CFTs suffer from the terminological nightmare that "bulk" and "boundary" have twofold meanings. To minimize confusion, we will mostly use "bulk" in the meaning of this sentence, *e.g.*, to refer to the CFT operators that lives in the full $\mathbb{R}^d$, to be contrasted to the interface or boundary operators that live at $x_\perp = 0$.

To proceed, we insert

$$\int_0^\infty d\rho\ \delta(\rho - \sum_i t_i - \sum_I s_I) = 1 \tag{3.68}$$

to replace all $\sum_i t_i + \sum_I s_I$ by ρ. Then we rescale t_i and s_I by $\rho^{1/2}$ so that all the powers of ρ are removed. Notice, after the rescaling, the only integral in ρ is the following delta function

$$\int_0^\infty d\rho\ \delta(\rho - \sqrt{\rho}(\sum_i t_i + \sum_I s_I)) = 2\ . \tag{3.69}$$

This turns the integral into

$$\begin{aligned} W_{n,m} = & \frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{\sum_i \Delta_i + \sum_I \widehat{\Delta}_I - d + 1}{2})}{\prod_i \Gamma(\Delta_i) \prod_I \Gamma(\widehat{\Delta}_I)} \int_0^\infty \prod_i \frac{dt_i}{t_i} t_i^{\Delta_i} \frac{ds_I}{s_I} s_I^{\widehat{\Delta}_I} \\ & \times \exp\left[-\sum_{i<j} t_i t_j (-2P_i \cdot P_j) - (\sum_i t_i P_i \cdot B)^2 - \sum_{I<J} s_I s_J (-2\widehat{P}_I \cdot \widehat{P}_J) - \sum_{i,I} t_i s_I (-2P_i \cdot \widehat{P}_I) \right] . \end{aligned} \tag{3.70}$$

We use the Mellin representation of exponential for the following terms in the exponent: all $t_i t_j$, all $s_I s_J$ and the $t_i s_I$ with $i > 1$. Their conjugate variables are respectively denoted as δ_{ij}, β_{IJ} and γ_{iI}. We then get the following integral

$$\begin{aligned} W_{n,m} = & \frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{\sum_i \Delta_i + \sum_I \widehat{\Delta}_I - d + 1}{2})}{\prod_i \Gamma(\Delta_i) \prod_I \Gamma(\widehat{\Delta}_I)} \int \prod_{i<j} \left([d\delta_{ij}] \Gamma(\delta_{ij}) (-2P_i \cdot P_j)^{-\delta_{ij}} \right) \\ & \times \int \prod_{I<J} \left([d\beta_{IJ}] \Gamma(\beta_{IJ}) (-2\widehat{P}_I \cdot \widehat{P}_J)^{-\beta_{IJ}} \right) \prod_{i>1,I} \left([d\gamma_{iI}] \Gamma(\gamma_{iI}) (-2P_i \cdot \widehat{P}_I)^{-\gamma_{iI}} \right) \\ & \times \int_0^\infty \prod_i \frac{dt_i}{t_i} t_i^{\Delta_i - \sum_{j\neq i} \delta_{ij} - \sum_I [1-\delta(i,1)]\gamma_{iI}} \int_0^\infty \prod_I \frac{ds_I}{s_I} s_I^{\widehat{\Delta}_I - \sum_{J\neq I} \beta_{IJ} - \sum_{i>1} \gamma_{iI}} \\ & \times \exp\left(-(\sum_i t_i P_i \cdot B)^2 \right) \exp\left(-\sum_I t_1 s_I (-2P_1 \cdot \widehat{P}_I) \right) . \end{aligned} \tag{3.71}$$

Here in the third line we denoted the Kronecker delta function as $\delta(a,b)$ in order to distinguish it from the Mellin variable δ_{ij}, and we hope that it will not cause any confusion to the reader. The s-integral can now be easily performed, giving

$$\prod_I \left[(-2P_1 \cdot \widehat{P}_I)^{-\widehat{\Delta}_I + \sum_{J\neq I} \beta_{IJ} + \sum_{i>1} \gamma_{iI}} \Gamma(\widehat{\Delta}_I - \sum_{J\neq I} \beta_{IJ} - \sum_{i>1} \gamma_{iI}) \right] \\ \times t_1^{-\sum_I (\widehat{\Delta}_I - \sum_{J\neq I} \beta_{IJ} - \sum_{i>1} \gamma_{iI})} . \tag{3.72}$$

We notice that the combination $\widehat{\Delta}_I - \sum_{J\neq I} \beta_{IJ} - \sum_{i>1} \gamma_{iI}$ is just γ_{1I} by (3.49). Plugging the s-integral result into the total integral, the t-integral just becomes

$$\int_0^\infty \prod_{i=1}^n \frac{dt_i}{t_i} t_i^{\Delta_i - \sum_{j\neq i} \delta_{ij} - \sum_I \gamma_{iI}} \exp\left(-(\sum_i t_i P_i \cdot B)^2 \right) \tag{3.73}$$

where we have used $\gamma_{1I} \equiv \widehat{\Delta}_I - \sum_{J\neq I} \beta_{IJ} - \sum_{i>1} \gamma_{iI}$. We can evaluate this integral by inserting

$$\int_0^\infty d\lambda\, \delta(\lambda - \sum_i t_i) = 1 , \tag{3.74}$$

and rescaling $t_i \to \lambda t_i$. Define $\alpha_i \equiv \Delta_i - \sum_{j\neq i} \delta_{ij} - \sum_I \gamma_{iI}$ as in (3.49), the new integral is

$$\int_0^\infty \prod_{i=1}^n \frac{dt_i}{t_i} t_i^{\alpha_i} \delta(1 - \sum_i t_i) \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{\sum_i \alpha_i} \exp\left(-\lambda^2 (\sum_i t_i P_i \cdot B)^2 \right) , \tag{3.75}$$

and it is not difficult to find that this integral evaluates to

$$\frac{1}{2} \left(\prod_i \Gamma(\alpha_i) (P_i \cdot B)^{-\alpha_i} \right) \frac{\Gamma(\frac{\sum_i \alpha_i}{2})}{\Gamma(\sum_i \alpha_i)} . \tag{3.76}$$

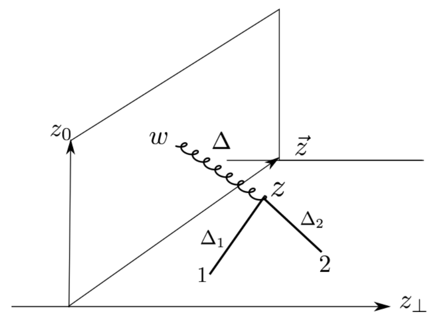

Figure 3.2: An exchange Witten diagram in the bulk channel.

All in all, we have obtained the following result for the general contact Witten diagram

$$
\begin{aligned}
W_{n,m} = & \frac{\pi^{\frac{d-1}{2}}\Gamma(\frac{\sum_i \Delta_i + \sum_I \widehat{\Delta}_I - d + 1}{2})}{2\prod_i \Gamma(\Delta_i)\prod_I \Gamma(\widehat{\Delta}_I)} \int \prod_{i<j} \Big([d\delta_{ij}]\Gamma(\delta_{ij})(-2P_i \cdot P_j)^{-\delta_{ij}} \Big) \\
& \times \int \prod_{I<J} \Big([d\beta_{IJ}]\Gamma(\beta_{IJ})(-2\widehat{P}_I \cdot \widehat{P}_J)^{-\beta_{IJ}} \Big) \prod_{iI} \Big([d\gamma_{iI}]\Gamma(\gamma_{iI})(-2P_i \cdot \widehat{P}_I)^{-\gamma_{iI}} \Big) \\
& \times \prod_i \big(d[\alpha_i]\Gamma(\alpha_i)(P_i \cdot B)^{-\alpha_i} \big) \frac{\Gamma(\frac{\sum_i \alpha_i}{2})}{\Gamma(\sum_i \alpha_i)}
\end{aligned}
\tag{3.77}
$$

where the integration variable are subject to the constraints (3.49).

Now let us extract the Mellin amplitude. Thanks to the relation

$$
\sum_i \alpha_i = \sum_i \Delta_i - \sum_{i,I} \gamma_{iI} - \sum_{i \neq j} \delta_{ij} = -\sum_{i,j} p_i \cdot p_j - \sum_{i,I} p_i \cdot \widehat{p}_I = -\mathcal{P}^2 \ , \tag{3.78}
$$

the outstanding ratio of Gamma functions therefore is just the one that appears in the definition (3.56). We thus find that such contact Witten diagrams all have constant Mellin amplitudes.

Bulk-Channel Exchange Witten Diagram

An exchange Witten diagram in the bulk channel is illustrated in Figure 3.2. We now evaluate this diagram using two different methods.

The first method uses the techniques from [36] to reduce an exchange Witten diagram to a finite sum of contact Witten diagrams. This method applies when the quantum numbers of the operators satisfy special relations.

The Witten diagram is given by the following integral,

$$W_{\rm bulk} = \int_{AdS_d} dW \int_{AdS_{d+1}} dZ \; G^{\Delta_1}_{B\partial}(P_1, Z)\; G^{\Delta_2}_{B\partial}(P_2, Z)\; G^{\Delta}_{BB}(Z, W)\;. \tag{3.79}$$

The Z-integral has been performed in [36] "without really trying". Let us briefly review that method. Denote the Z-integral as

$$A(W, P_1, P_2) = \int_{AdS_{d+1}} dZ \; G^{\Delta_1}_{B\partial}(P_1, Z)\; G^{\Delta_2}_{B\partial}(P_2, Z)\; G^{\Delta}_{BB}(Z, W). \tag{3.80}$$

It is convenient to perform a translation such that

$$x_1 \to 0\;, \quad x_2 \to x_{21} \equiv x_2 - x_1\;. \tag{3.81}$$

This is followed by a conformal inversion,

$$x'_{12} = \frac{x_{12}}{(x_{12})^2}\;, \quad z' = \frac{z}{z^2}\;, \quad w' = \frac{2}{w^2}. \tag{3.82}$$

After these transformations the integral becomes,

$$A(W, P_1, P_2) = (x_{12})^{-2\Delta_2} I(w' - x'_{12}) \tag{3.83}$$

where

$$I(w) = \int \frac{d^{d+1}z}{z_0^{d+1}}\; G^{\Delta}_{BB}(-2Z\cdot W)\; z_0^{\Delta_1} \left(\frac{z_0}{z^2}\right)^{\Delta_2}\;. \tag{3.84}$$

The scaling behavior of $I(w)$ under $w \to \lambda w$ together with the Poincaré symmetry dictates that $I(w)$ takes the form

$$I(w) = w_0^{\Delta_1 - \Delta_2} f(t) \tag{3.85}$$

where

$$t = \frac{w_0^2}{w^2} \tag{3.86}$$

On the other hand $f(t)$ is constrained by the following differential equation,

$$\begin{aligned} 4t^2(t-1)f'' + 4t[(\Delta_1 - \Delta_2 + 1)t - \Delta_1 + \Delta_3 + \frac{d}{2} - 1]f' \\ + [(\Delta_1 - \Delta_2)(d - \Delta_1 + \Delta_2) + m^2]f = t^{\Delta_2} \end{aligned} \tag{3.87}$$

where $m^2 = \Delta(\Delta - d)$. This equation comes from acting with the equation of motion of the field in the bulk-to-bulk propagator,

$$-\Box_{AdS_{d+1},W} G^{\Delta}_{BB}(Z,W) + m^2 G^{\Delta}_{BB}(Z,W) = \delta(Z,W) \tag{3.88}$$

and it collapses the bulk-to-bulk propagator to a delta-function. The solution to this equation is generically hypergeometric functions of type ${}_2F_1$ which expands to an infinite series, however with appropriate choice of conformal dimensions, $f(t)$ admits a polynomial solution:

$$f(t) = \sum_{k=k_{\min}}^{k_{\max}} a_k t^k \tag{3.89}$$

with

$$\begin{aligned} &k_{\min} = (\Delta - \Delta_1 + \Delta_2)/2 \ , \qquad k_{\max} = \Delta_2 - 1 \ , \\ &a_{k-1} = a_k \frac{(k - \frac{\Delta}{2} + \frac{\Delta_1 - \Delta_2}{2})(k - \frac{d}{2} + \frac{\Delta}{2} + \frac{\Delta_1 - \Delta_2}{2})}{(k-1)(k-1-\Delta_1+\Delta_2)} \ , \\ &a_{\Delta_2 - 1} = \frac{1}{4(\Delta_1 - 1)(\Delta_2 - 1)} \end{aligned} \tag{3.90}$$

and this truncation happens when $\Delta_1 + \Delta_2 - \Delta$ is a positive even integer. After obtaining this solution, we can undo the inversion and translation and the upshot is that $A(W, P_1, P_2)$ becomes a sum of contact vertices at W,

$$A(W, P_1, P_2) = \sum_{k=k_{\min}}^{k_{\max}} a_k (-2P_1 \cdot P_2)^{k-\Delta_2} G^{k+\Delta_1-\Delta_2}_{B\partial}(P_1, W) \ G^{k}_{B\partial}(P_2, W) \tag{3.91}$$

This identity is illustrated in Figure 3.3.

We can then use the formula for two-point contact Witten diagrams (which is a special case of (3.77)) to obtain the Mellin amplitude of a bulk exchange Witten diagram,

$$\mathcal{M}_{\text{bulk}} = \pi^{d/2} \sum_{k_{\min}}^{k_{\max}} \frac{a_k \Gamma(\frac{\Delta_1 - \Delta_2 + 2k - (d-1)}{2})}{\Gamma(\Delta_1 - \Delta_2 + k)\Gamma(k)} \frac{\Gamma(\tau + k - \Delta_2)}{\Gamma(\tau)} \ . \tag{3.92}$$

This Mellin amplitude has finitely many simple poles in τ. Alternatively in terms of the squared interface momentum $\mathcal{P}^2 = 2\tau - \Delta_1 - \Delta_2$, the simple poles of the Mellin amplitude are located at

$$-\Delta, -\Delta - 2, \ldots, -\Delta_1 - \Delta_2 + 2 \tag{3.93}$$

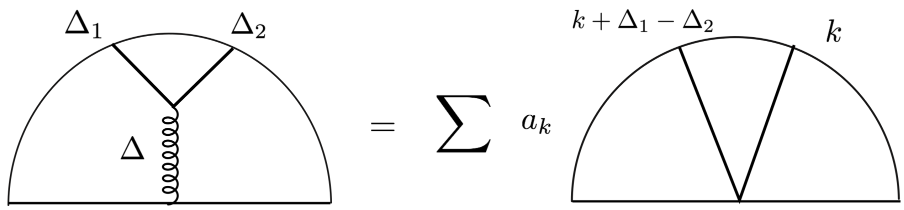

Figure 3.3: A bulk exchange Witten diagram is replaced by a sum of contact Witten diagrams when $\Delta_1+\Delta_2-\Delta$ is a positive even integer.

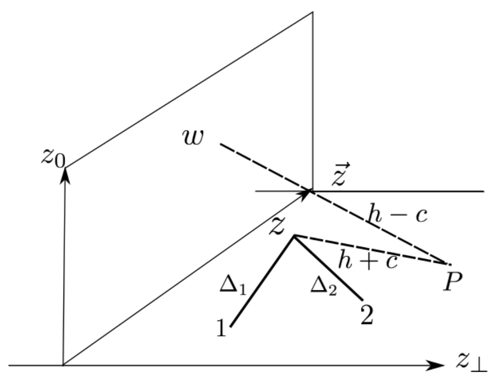

Figure 3.4: Using the split representation of the bulk-to-bulk propagator the bulk exchange Witten diagram is reduced to the product of a three-point contact Witten diagram and an one-point contact Witten diagram.

resembling a resonance amplitude with intermediate particles whose squared masses are $\Delta, \Delta+2, \ldots, \Delta_1+\Delta_2-2$.

The second method exploits the split representation of the bulk-to-bulk propagator and applies to diagrams with general quantum numbers.

To begin, we use the spectral representation of the bulk-to-bulk propagator [41],

$$G_{BB}^{\Delta}(Z,W)=\int_{-i\infty}^{i\infty}\frac{dc}{(\Delta-h)^2-c^2}\frac{\Gamma(h+c)\Gamma(h-c)}{2\pi^{2h}\Gamma(c)\Gamma(-c)}\int dP(-2P\cdot Z)^{h+c}(-2P\cdot W)^{h-c}\,, \tag{3.94}$$

where $h=\frac{d}{2}$. Using this representation, the exchange Witten diagram is written as a product of three-point contact Witten diagram in AdS_{d+1} and an one-point function. There is a common point P sitting on the boundary of AdS_{d+1} which is integrated over. This is schematically represented by Figure

3.4.

Explicitly, denoting the three-point function by $\langle O_{\Delta_1}(P_1)O_{\Delta_2}(P_2)O_{h+c}(P)\rangle$ and the one-point function by $\langle O_{h-c}(P)\rangle$, the Witten diagram is given by

$$W_{\rm bulk} = \int dP \int dc \frac{\langle O_{\Delta_1}(P_1)O_{\Delta_2}(P_2)O_{h+c}(P)\rangle\langle O_{h-c}(P)\rangle}{(\Delta-h)^2-c^2}\frac{\Gamma(h+c)\Gamma(h-c)}{2\pi^{2h}\Gamma(c)\Gamma(-c)}\,. \tag{3.95}$$

The three-point Witten diagram can be easily evaluated,

$$\begin{aligned}\langle O_{\Delta_1}(P_1)O_{\Delta_2}(P_2)O_{h+c}(P)\rangle &= \frac{\pi^h}{2}\frac{\Gamma(\frac{\Delta_1+\Delta_2+h+c}{2}-2)}{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(h+c)} \\ &\times \frac{\Gamma(\frac{\Delta_1-\Delta_2+h+c}{2})\Gamma(\frac{\Delta_1+\Delta_2-h-c}{2})\Gamma(\frac{-\Delta_1+\Delta_2+h+c}{2})}{(-P_1\cdot P_2)^{\Delta_2+\Delta_1-h-c}(-P_2\cdot P)^{h+c+\Delta_2-\Delta_1}(-2P_1\cdot P)^{\Delta_1-\Delta_2+h+c}}\,.\end{aligned} \tag{3.96}$$

The one-point function is

$$\langle O_{h-c}(P)\rangle = \frac{\pi^{h-\frac{1}{2}}}{2}\frac{\Gamma(\frac{h-c}{2})\Gamma(\frac{-h-c+1}{2})}{\Gamma(h-c)}\frac{1}{(P\cdot B)^{h-c}}\,. \tag{3.97}$$

Then the only non-trivial integral remains to be evaluated is the P-integral

$$\begin{aligned}&\int_{-\infty}^{+\infty} dp_\perp \int_{-\infty}^{+\infty} d\vec{p}\frac{1}{(x_2-p)^{h+c+\Delta_2-\Delta_1}}\frac{1}{(x_1-p)^{h+c+\Delta_1-\Delta_2}}\frac{1}{p_\perp^{h-c}} \\ &= \frac{\pi^h}{\Gamma(\frac{h+c+\Delta_1-\Delta_2}{2})\Gamma(\frac{h+c+\Delta_2-\Delta_1}{2})\Gamma(\frac{h-c}{2})}\int \frac{ds}{s}\frac{dt}{t}\frac{du}{u}s^{\frac{h-c}{2}}t^{\frac{h+c+\Delta_1-\Delta_2}{2}}u^{\frac{h+c+\Delta_2-\Delta_1}{2}} \\ &\times (u+t)^{-h+\frac{1}{2}}(s+t+u)^{-\frac{1}{2}}\times\exp\left(-\frac{tu(x-y)^2}{t+u}-\frac{s(tx_\perp+uy_\perp)^2}{(t+u)(s+t+u)}\right)\,.\end{aligned} \tag{3.98}$$

We insert into the integral the identity

$$1 = \int d\lambda\,\delta(\lambda-(s+t+u))\int d\rho\,\delta(\rho-(t+u)) \tag{3.99}$$

and rescale first $t\to\rho t$, $u\to\rho u$, followed by $\lambda\to\lambda\rho$ and the use of an inverse Mellin transformation on the $(tx_\perp+uy_\perp)^2$ exponent. The integrals then become elementary. The final result for the bulk exchange Witten

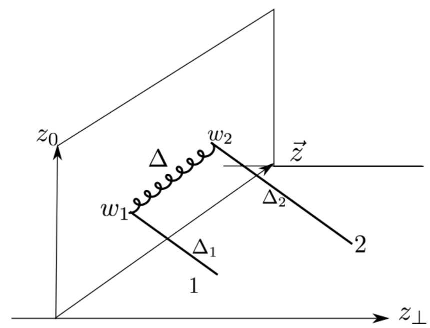

Figure 3.5: An exchange Witten diagram in the interface channel.

diagram is

$$W_{\text{bulk}} = \frac{\pi^{h-\frac{1}{2}}}{2\Gamma(\Delta_1)\Gamma(\Delta_2)x_\perp^{\Delta_1}y_\perp^{\Delta_2}} \int_{-i\infty}^{+i\infty} dc \int_{-i\infty}^{+i\infty} d\tau \frac{f(c,\tau)f(-c,\tau)}{(\Delta-h)^2-c^2}\eta^{\tau-\frac{\Delta_1+\Delta_2}{2}}$$
$$\times \frac{\Gamma(\tau)\Gamma(\tau+\frac{\Delta_1-\Delta_2}{2})\Gamma(\tau+\frac{\Delta_2-\Delta_1}{2})}{\Gamma(\frac{1}{2}-\tau)\Gamma(2\tau)}, \tag{3.100}$$

where $h = d/2$ and

$$f(c,\tau) = \frac{\Gamma(\frac{\Delta_1+\Delta_2-h+c}{2})\Gamma(\frac{1+c-h}{2})\Gamma(\frac{h+c}{2}-\tau)}{2\Gamma(c)}\ . \tag{3.101}$$

Interface-Channel Exchange Witten Diagram

An exchange Witten diagram in the interface channel is illustrated in Figure 3.5. Similarly this diagram can also be evaluated using two methods.

We start first with the truncation methods assuming the quantum numbers of the operators are fine-tuned to satisfy special relations. The interface exchange Witten diagram represented by Figure 3.5 is given by the following integral,

$$W_{\text{interface}} = \int_{AdS_d} dW_1 dW_2\ G_{B\partial}^{\Delta_1}(P_1,W_1)\ G_{BB,\ AdS_d}^{\Delta}(W_1,W_2)\ G_{B\partial}^{\Delta_2}(P_2,W_2)\ . \tag{3.102}$$

We focus on the integral of W_1 denoted as

$$A(P_1,W_2) = \int_{AdS_d} dW_1\ G_{B\partial}^{\Delta_1}(P_1,W_1)\ G_{BB,\ AdS_d}^{\Delta}(W_1,W_2)\ . \tag{3.103}$$

This integral has AdS_d isometry and should depend on a single variable t invariant under the scaling $w_2 \to \lambda w_2$, $x_1 \to \lambda x_1$

$$t \equiv \frac{P_1 \cdot W_2}{P_1 \cdot B} = \frac{w_{2,0}^2 + x_{1,\perp}^2 + (\vec{w}_2 - \vec{x}_1)^2}{w_{2,0} x_{1,\perp}} \ . \tag{3.104}$$

The function $A(P_1, W_2)$ therefore takes the form

$$A(P_1, W_2) = x_{1,\perp}^{-\Delta_1} f(t) \ . \tag{3.105}$$

To work out $f(t)$, we use the equation of motion for the bulk-to-bulk propagator inside AdS_d. It leads to the following equation

$$(-\Box_{W_2, AdS_d} + m^2)(x_{1,\perp}^{-\Delta_1} f(t)) = x_{1,\perp}^{-\Delta_1} t^{-\Delta_1} \tag{3.106}$$

where $m^2 = \Delta(\Delta - (d-1))$. The Laplacian acts on a function of t as

$$\Box_{W_2, AdS_d} f(t) = (t^2 - 4) f''(t) + dt f'(t) \ . \tag{3.107}$$

The function $f(t)$ also admits a polynomial solution when $\Delta < \Delta_1$ and has even integer difference,

$$f(t) = \sum_{k_{\min}}^{k_{\max}} a_k t^k \ , \tag{3.108}$$

where

$$\begin{aligned} a_{k+2} &= \frac{(k+\Delta)(k-(\Delta-(d-1)))}{4(k+1)(k+2)} a_k \ , \\ k_{\min} &= -\Delta_1 + 2 \ , \\ k_{\max} &= -\Delta \ , \\ a_{k_{\min}} &= \frac{1}{4(-\Delta_1+2)(-\Delta_1+1)} \ . \end{aligned} \tag{3.109}$$

Using the definition of t in (3.104), we find that each monomial of t corresponds to a contact vertex. The polynomial solution to $f(t)$ means we can express the exchange Witten diagram as a sum of contact Witten diagrams (Figure 3.6),

$$W_{\text{interface}} = \int_{AdS_d} dW_2 \sum_{k_{\min}}^{k_{\max}} a_k x_{1,\perp}^{-\Delta_1 - k} \ G_{B\partial}^{-k}(P_1, W_2) \ G_{B\partial}^{\Delta_2}(P_2, W_2) \ . \tag{3.110}$$

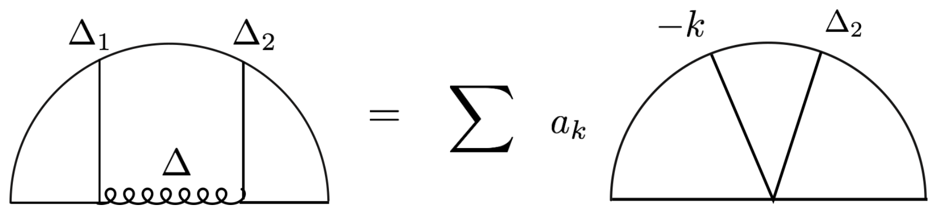

Figure 3.6: The interface exchange Witten diagram is replaced by a finite sum of contact Witten diagrams when $\Delta_1 - \Delta$ is a positive even integer.

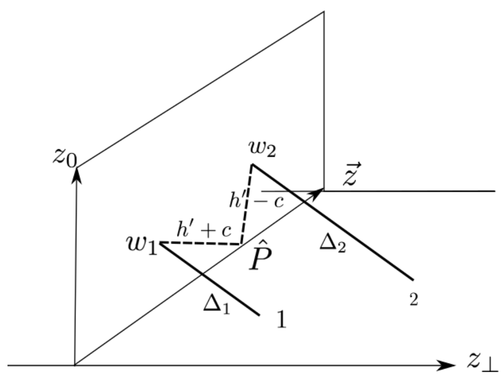

Figure 3.7: Using the split representation of the bulk-to-bulk propagator the interface exchange Witten diagram is reduced to the product of two bulk-interface two-point contact Witten diagrams.

Using the Mellin formula for contact Witten diagrams (3.77), we get the Mellin amplitude for the interface exchange Witten diagram

$$\mathcal{M}_{\text{interface}} = \pi^{\frac{d}{2}} \sum_{k_{\min}}^{k_{\max}} a_k 2^{k+\Delta_1} \frac{\Gamma(\frac{\Delta_2 - k - (d-1)}{2})}{\Gamma(-k)\Gamma(\Delta_2))} \times \frac{\Gamma(-k-\tau)\Gamma(\frac{\Delta_1+\Delta_2+1}{2} - \tau)}{\Gamma(\Delta_1 - \tau)\Gamma(\frac{-k+\Delta_2+1}{2} - \tau)} . \tag{3.111}$$

When the operators have generic quantum numbers, this method no longer applies. To evaluate these exchange Witten diagrams, we use a similar method exploiting the spectral representation of the bulk-to-bulk propagator in AdS_d.

The AdS_d propagator can be written as

$$G_{BB}^{\Delta,AdS_d} = \int_{i\infty}^{i\infty} \frac{dc}{(\Delta-h')^2-c^2} \frac{\Gamma(h'+c)\Gamma(h'-c)}{2\pi^{2h'}\Gamma(c)\Gamma(-c)} \int d\widehat{P}(-2\widehat{P}\cdot W_1)^{h'+c}(-2\widehat{P}\cdot W_2)^{h'-c} \tag{3.112}$$

with $h' = (d-1)/2$ to write the Witten diagram into the product of two bulk-interface two-point functions $W_{1,1}$. This splitting is schematically illustrated in Figure 3.7. Notice the point $\widehat{P}$ being integrated over is sitting at the boundary of AdS_d. Denoting the two-point functions by $\langle O_{\Delta_1}(P_1)\widehat{O}_{h'+c}(\widehat{P})\rangle$ and $\langle O_{\Delta_2}(P_2)\widehat{O}_{h'-c}(\widehat{P})\rangle$, the Witten diagram now takes the form

$$W_{\text{interface}} = \int d\widehat{P} \int dc \frac{\langle O_{\Delta_1}(P_1)\widehat{O}_{h'+c}(\widehat{P})\langle O_{\Delta_2}(P_2)\widehat{O}_{h'-c}(\widehat{P})\rangle}{(\Delta-h')^2-c^2} \frac{\Gamma(h'+c)\Gamma(h'-c)}{2\pi^{2h'}\Gamma(c)\Gamma(-c)} . \tag{3.113}$$

The two-point functions have been worked out as a special case of (3.77). The only integral we need to do is the $\widehat{P}$-integral

$$\int d^{d-1}\vec{p}\, (x_{1,\perp}^2 + (\vec{x}_1-\vec{p})^2)^{-(h'+c)}(x_{2,\perp}^2 + (\vec{x}_2-\vec{p})^2)^{-(h'-c)} . \tag{3.114}$$

Evaluating this integral presents little difficulty using the techniques we have developed in the previous sections. The answer is simply

$$\frac{\pi^{h'}}{\Gamma(h'+c)\Gamma(h'-c)} \int_{-i\infty}^{i\infty} d\tau \eta^{-\tau} \frac{\Gamma(\tau)\Gamma(h'-\tau)\Gamma(h'+c-\tau)\Gamma(h'-c-\tau)}{(x_{1,\perp})^{h'+c}(x_{2,\perp})^{h'-c}\Gamma(2h'-2\tau)} . \tag{3.115}$$

Hence the interface exchange Witten diagram is given by the following spectral representation

$$\begin{aligned} W_{\text{interface}} = & \frac{\pi^{h'}}{2\Gamma(\Delta_1)\Gamma(\Delta_2)x_\perp^{\Delta_1}y_\perp^{\Delta_2}} \int_{-i\infty}^{+i\infty} d\tau \eta^{-\tau} \frac{\Gamma(\tau)\Gamma(h'-\tau)}{\Gamma(2h'-2\tau)} \\ & \times \int_{-i\infty}^{+i\infty} dc \frac{1}{(\Delta-h')^2-c^2} f(c,\tau)f(-c,\tau) \end{aligned} \tag{3.116}$$

where $h' = (d-1)/2$ and

$$f(c,\tau) = \frac{\Gamma(h'+c-\tau)\Gamma(\frac{\Delta_1-h'+c}{2})\Gamma(\frac{\Delta_2-h'+c}{2})}{2\Gamma(c)} . \tag{3.117}$$

Chapter 4

An Algebraic Bootstrap Problem in Mellin Space

While simpler than the standard perturbative recipe, the position space method in Section 2.3 also quickly runs out of steam as the KK level is increased. What's worse, the answer takes a completely unintuitive form, with no simple general pattern. On the other hand, the Witten diagrams admit significantly simpler representation in Mellin space compared to the position space. Their simple analytic structure gives us more control in the analysis. In this Chapter and the subsequent Chapter 5, we take advantage of this simplicity and introduce two complimentary methods to compute four-point functions in Mellin space.

The method that we will introduce in this section is based on the idea of bootstrap: we view the task of computing four-point functions as solving an algebraic problem formulated by imposing symmetry constraints and self-consistency conditions. More precisely, tree-level holographic correlators in $AdS_5 \times S^5$ and $AdS_7 \times S^4$ are rational functions of Mandelstam-like invariants, with poles and residues controlled by OPE factorization, in close analogy with tree-level flat space scattering amplitudes. Additionally, the amplitude enjoys Bose symmetry and has well-behaved asymptotic limit. Superconformal symmetry is made manifest by solving the superconformal Ward identity in terms of an "auxiliary" Mellin amplitude. The consistency conditions that this amplitude must satisfy define a very constrained algebraic problem, which very plausibly admits a unique solution. While the position space method is implemented on a case-by-case basis for different correlators, the Mellin algebraic problem takes a universal form. We were able to solve the problem in one fell swoop for all half-BPS four-point function of $AdS_5 \times S^5$ with arbitrary weights – a feat extremely difficult to replicate in position space.

In Section 4.1 and Section 4.3, we set up the algebraic bootstrap problems for $AdS_5 \times S^5$ and $AdS_7 \times S^4$. A key step is to translate the position space solution of the superconformal Ward identity into Mellin space. Though the strategy is the same, the AdS_7 case is significantly more involved. In Section 4.2 we present the complete solution for one-half BPS four-point functions in $AdS_5 \times S^5$ with arbitrary KK modes. For $AdS_7 \times S^4$, we have not yet been able to obtain the general solution. We present partial results in Section 4.4.

4.1 Formulating an Algebraic Bootstrap Problem: $AdS_5 \times S^5$

4.1.1 Rewriting the Superconformal Ward Identity

In four dimensions, $\epsilon = 2$ and the superconformal Ward identity (2.23) can be easily solved in position space. The solution is [39, 38][1]

$$\mathcal{G}(U,V;\sigma,\tau) = \mathcal{G}_{\text{free}}(U,V;\sigma,\tau) + R\,\mathcal{H}(U,V;\sigma,\tau)\,, \tag{4.1}$$

where $\mathcal{G}_{\text{free}}$ is the answer in free SYM theory and

$$\begin{aligned} R &= \tau\,1 + (1-\sigma-\tau)\,V + (-\tau-\sigma\tau+\tau^2)\,U + (\sigma^2-\sigma-\sigma\tau)\,UV \\ &\quad +\sigma V^2 + \sigma\tau\,U^2 \\ &= (1-z\alpha)(1-\bar{z}\alpha)(1-z\bar{\alpha})(1-\bar{z}\bar{\alpha})\,. \end{aligned} \tag{4.2}$$

All dynamical information is contained in the *a priori* unknown function $\mathcal{H}(U,V;\sigma,\tau)$. Note that $\mathcal{H}(U,V;\sigma,\tau)$ is a polynomial in σ, τ of degree $\mathcal{L}-2$.

We now turn to analyze the constraints of superconformal symmetry on the Mellin amplitude by translating the above solution into Mellin space. We rewrite (4.1) for the connected correlator,

$$\mathcal{G}_{\text{conn}}(U,V;\sigma,\tau) = \mathcal{G}_{\text{free,conn}}(U,V;\sigma,\tau) + R(U,V;\sigma,\tau)\,\mathcal{H}(U,V;\sigma,\tau)\;, \tag{4.3}$$

[1]There is an implicit regularity assumption for $\mathcal{H}(U,V;\sigma,\tau)$ as $\bar{\alpha} \to 1/\bar{z}$, otherwise the following equation would be an empty statement.

and take the Mellin transform of both sides of this equation. The transform[2] of the left-hand side gives the reduced Mellin amplitude M,

$$M(s,t;\sigma,\tau)=\int_0^\infty dUdV U^{\frac{-s+k_3+k_4-2\mathcal{L}-2}{2}} V^{\frac{-t+\min\{k_1+k_4,k_2+k_3\}-2}{2}} \mathcal{G}_{\rm conn}(U,V;\sigma,\tau)\ , \tag{4.4}$$

from which we define the Mellin amplitude $\mathcal{M}$,

$$\mathcal{M}(s,t;\sigma,\tau)\equiv\frac{M(s,t;\sigma,\tau)}{\Gamma_{k_1k_2k_3k_4}}\ , \tag{4.5}$$

where

$$\begin{aligned}\Gamma_{k_1k_2k_3k_4}\equiv\,&\Gamma[-\frac{s}{2}+\frac{k_1+k_2}{2}]\Gamma[-\frac{s}{2}+\frac{k_3+k_4}{2}]\Gamma[-\frac{t}{2}+\frac{k_2+k_3}{2}]\\ &\times\Gamma[-\frac{t}{2}+\frac{k_1+k_4}{2}]\Gamma[-\frac{u}{2}+\frac{k_1+k_3}{2}]\Gamma[-\frac{u}{2}+\frac{k_2+k_4}{2}]\ ,\\ &u\equiv k_1+k_2+k_3+k_4-s-t\,.\end{aligned} \tag{4.6}$$

On the right-hand side of (4.3), the first term is the free part of the correlator. It consists of a sum of terms of the form $\sigma^a\tau^bU^mV^n$, where m, n are integers and a, b non-negative integers. The Mellin transform of any such term is ill-defined. As we shall explain in Section 4.1.3, there is a consistent sense in which it can be defined to be *zero*. The function $\mathcal{G}_{\rm free,conn}(U,V;\sigma,\tau)$ will be recovered as a regularization effect in transforming back from Mellin space to position space.

We then turn to the second term on the on the right-hand side of (4.3). We define an auxiliary amplitude $\widetilde{\mathcal{M}}$ from the Mellin transform of the dynamical function $\mathcal{H}$,

$$\widetilde{\mathcal{M}}(s,t;\sigma,\tau)=\frac{\int_0^\infty dUdV U^{\frac{-s+k_3+k_4-2\mathcal{L}-2}{2}} V^{\frac{-t+\min\{k_1+k_4,k_2+k_3\}-2}{2}} \mathcal{H}(U,V;\sigma,\tau)}{\tilde{\Gamma}_{k_1k_2k_3k_4}}\ , \tag{4.7}$$

[2]This definition should be taken with a grain of salt. In general, the integral transform of the full connected correlator is divergent. In the supergravity limit, there is a natural decomposition of $\mathcal{G}_{\rm conn}$ into a sum of $\bar{D}$ functions, each of which has a well-defined Mellin transform in a certain region of the s and t complex domains. However, it is often the case that there is no common region such that the transforms of the $\bar{D}$ functions are all convergent. On the other hand, the *inverse* Mellin transform (4.10) is well-defined, but care must be taken in specifying the integration contours. We will come back to this subtlety in Section 4.1.3.

with

$$\tilde{\Gamma}_{k_1k_2k_3k_4} \equiv \Gamma[-\frac{s}{2}+\frac{k_1+k_2}{2}]\Gamma[-\frac{s}{2}+\frac{k_3+k_4}{2}]\Gamma[-\frac{t}{2}+\frac{k_2+k_3}{2}] \\ \times\Gamma[-\frac{t}{2}+\frac{k_1+k_4}{2}]\Gamma[-\frac{\tilde{u}}{2}+\frac{k_1+k_3}{2}]\Gamma[-\frac{\tilde{u}}{2}+\frac{k_2+k_4}{2}]\,. \tag{4.8}$$

Note that we have introduced a "shifted" Mandelstam variable $\tilde{u}$,

$$\tilde{u} \equiv u-4 = k_1+k_2+k_3+k_4-4-s-t\,. \tag{4.9}$$

This shift is motived by the desire to keep the crossing symmetry properties of $\mathcal{H}$ as simple as possible, as we shall explain shortly. Let us also record the expressions of the inverse transforms,

$$\mathcal{G}_{\rm conn} = \int \frac{ds}{2}\frac{dt}{2}\, U^{\frac{s-k_3-k_4}{2}+\mathcal{L}} V^{\frac{t-\min\{k_1+k_4,k_2+k_3\}}{2}} \mathcal{M}(s,t;\sigma,\tau)\Gamma_{k_1k_2k_3k_4} \tag{4.10}$$

$$\mathcal{H} = \int \frac{ds}{2}\frac{dt}{2} U^{\frac{s-k_3-k_4}{2}+\mathcal{L}} V^{\frac{t-\min\{k_1+k_4,k_2+k_3\}}{2}} \widetilde{\mathcal{M}}(s,t;\sigma,\tau)\tilde{\Gamma}_{k_1k_2k_3k_4}\,, \tag{4.11}$$

where the precise definition of the integration contours will require a careful discussion in Section 4.1.3 below.

We are now ready to write down the Mellin translation of (4.3). It takes the simple form

$$\mathcal{M}(s,t;\sigma,\tau) = \widehat{R}\circ\widetilde{\mathcal{M}}(s,t,;\sigma,\tau)\,. \tag{4.12}$$

The multiplicative factor R has turned into a *difference operator* $\widehat{R}$,

$$\widehat{R} = \tau\,1+(1-\sigma-\tau)\,\widehat{V}+(-\tau-\sigma\tau+\tau^2)\,\widehat{U}+(\sigma^2-\sigma-\sigma\tau)\,\widehat{UV}+\sigma\widehat{V^2}+\sigma\tau\,\widehat{U^2}\,, \tag{4.13}$$

where the hatted monomials in U and V are defined to act as follows,

$$\widehat{U^mV^n}\circ\widetilde{\mathcal{M}}(s,t;\sigma,\tau) \equiv \widetilde{\mathcal{M}}(s-2m,t-2n);\sigma,\tau)\times\left(\frac{k_2+k_4-u}{2}\right)_{2-m-n} \\ \times\left(\frac{k_1+k_2-s}{2}\right)_m\left(\frac{k_3+k_4-s}{2}\right)_m\left(\frac{k_2+k_3-t}{2}\right)_n \\ \times\left(\frac{k_1+k_4-t}{2}\right)_n\left(\frac{k_1+k_3-u}{2}\right)_{2-m-n}\,, \tag{4.14}$$

with $(a)_n \equiv \Gamma[a+n]/\Gamma[a]$ the usual Pochhammer symbol.

Crossing symmetry and $\tilde{u}$

The Mellin amplitude $\mathcal{M}$ satisfies Bose symmetry, namely, it is invariant under permutation of the Mandelstam variables s, t, u if the external quantum numbers are also permuted accordingly. The auxiliary amplitude $\widetilde{\mathcal{M}}$ has been defined to enjoy the same symmetry under permutation of the *shifted* Mandelstam variables s, t, $\tilde{u}$. The point is that the factor R multiplying $\mathcal{H}$ is not crossing-invariant, and the shift in u precisely compensates for this asymmetry. Let us see this in some detail.

To make expressions more compact, we introduce some shorthand notations for the following combinations of coordinates,

$$\begin{aligned} A &= x_{12}^2 x_{34}^2\ , \quad & B &= x_{13}^2 x_{24}^2\ , \quad & C &= x_{14}^2 x_{23}^2\ , \\ a &= t_{12} t_{34}\ , \quad & b &= t_{13} t_{24}\ , \quad & c &= t_{14} t_{23}\ . \end{aligned} \tag{4.15}$$

In the equal-weights case (on which we focus for simplicity), the four-point function $G(x_i, t_i)$ is related to $\mathcal{G}(U, V; \sigma, \tau)$ by

$$G(x_i, t_i) = \left(\frac{a}{A}\right)^{\mathcal{L}} \mathcal{G}(U, V; \sigma, \tau)\ . \tag{4.16}$$

Substituting into this expression the inverse Mellin transformation (4.10), one finds

$$\begin{aligned} G(x_i, t_i) = \int_{i\infty}^{i\infty} ds dt \sum_{I+J+K=\mathcal{L}} & A^{\frac{s}{2}-\mathcal{L}} B^{\frac{u}{2}-\mathcal{L}} C^{\frac{t}{2}-\mathcal{L}} a^K b^I c^J \mathcal{M}_{IJK}(s,t) \\ & \times \Gamma^2[-\frac{s}{2}+\mathcal{L}] \Gamma^2[-\frac{t}{2}+\mathcal{L}] \Gamma^2[-\frac{u}{2}+\mathcal{L}]\ , \end{aligned} \tag{4.17}$$

where we defined $\sum_{I+J+K=\mathcal{L}} a^K b^I c^J \mathcal{M}_{IJK}(s,t) \equiv a^{\mathcal{L}} \mathcal{M}(s, t; \sigma, \tau)$. In terms of these new variables, crossing amounts to permuting simultaneously (A, B, C) and (a, b, c):

$$\begin{aligned} 1 \leftrightarrow 4: \quad & \left\{ \begin{array}{c} \sigma \leftrightarrow 1/\sigma,\ \tau \leftrightarrow \sigma/\tau, \\ U \leftrightarrow 1/U,\ V \leftrightarrow V/U \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{c} A \leftrightarrow B \\ a \leftrightarrow b \end{array} \right\}\ , \\ 1 \leftrightarrow 3: \quad & \left\{ \begin{array}{c} \sigma \leftrightarrow \sigma/\tau,\ \tau \leftrightarrow 1/\tau, \\ U \leftrightarrow V,\ V \leftrightarrow U \end{array} \right\} \quad \text{or} \quad \left\{ \begin{array}{c} A \leftrightarrow C \\ a \leftrightarrow c \end{array} \right\}\ . \end{aligned} \tag{4.18}$$

Invariance of the four-point function under crossing implies that the Mellin amplitude $\mathcal{M}(s, t; \sigma, \tau)$ must obey

$$\begin{aligned} \sigma^{\mathcal{L}} \mathcal{M}(u, t; 1/\sigma, \tau/\sigma) &= \mathcal{M}(s, t; \sigma, \tau)\ , \\ \tau^{\mathcal{L}} \mathcal{M}(t, s; \sigma/\tau, 1/\tau) &= \mathcal{M}(s, t; \sigma, \tau)\ . \end{aligned} \tag{4.19}$$

On the other hand, a similar representation exists for $R\mathcal{H}$. The factor R can be expressed as

$$
\begin{aligned}
R = \frac{1}{a^2B^2}\big(&a^2BC + b^2AC + c^2AB - abAC - abBC + abC^2 \\
&- acAB + acB^2 - acBC + bcA^2 - bcAB - bcAC\big) \\
\equiv \frac{\mathfrak{R}}{a^2B^2}\,,&
\end{aligned}
\tag{4.20}
$$

with a crossing-invariant numerator $\mathfrak{R}$ but a non-invariant denominator. When we go to the Mellin representation of $\left(\frac{a}{A}\right)^{\mathcal{L}} R\mathcal{H}$ by substituting in (4.11), we find that the power of B receives an additional -2 from the denominator of R in (4.20), explaining the shift from u to $\tilde{u}$,

$$
\begin{aligned}
\left(\frac{a}{A}\right)^{\mathcal{L}} R\mathcal{H} = \int_{i\infty}^{i\infty} dsdt \sum_{i+j+k=\mathcal{L}-2} A^{\frac{s}{2}-\mathcal{L}} B^{\frac{\tilde{u}}{2}-\mathcal{L}} C^{\frac{t}{2}-\mathcal{L}} a^k\, b^i\, c^j\, \mathfrak{R}\, \widetilde{\mathcal{M}}_{ijk}(s,t) \\
\times \Gamma^2[-\frac{s}{2}+\mathcal{L}]\Gamma^2[-\frac{t}{2}+\mathcal{L}]\Gamma^2[-\frac{\tilde{u}}{2}+\mathcal{L}]\,.
\end{aligned}
\tag{4.21}
$$

Here we have similarly defined

$$
\sum_{i+j+k=\mathcal{L}-2} a^k\, b^i\, c^j\, \mathfrak{R}\, \widetilde{\mathcal{M}}_{ijk}(s,t) = a^{\mathcal{L}-2}\widetilde{\mathcal{M}}(s,t;\sigma,\tau)\,. \tag{4.22}
$$

Invariance of this expression under crossing implies the following transformation rules for $\widetilde{\mathcal{M}}$,

$$
\begin{aligned}
\sigma^{\mathcal{L}-2}\widetilde{\mathcal{M}}(\tilde{u},t;1/\sigma,\tau/\sigma) = \widetilde{\mathcal{M}}(s,t,;\sigma,\tau)\,, \\
\tau^{\mathcal{L}-2}\widetilde{\mathcal{M}}(t,s;\sigma/\tau,1/\tau) = \widetilde{\mathcal{M}}(s,t;\sigma,\tau)\,.
\end{aligned}
\tag{4.23}
$$

We see that in the auxiliary amplitude $\widetilde{\mathcal{M}}$, the role of u is played by $\tilde{u}$. This generalizes to the unequal-weight cases.

4.1.2 An Algebraic Problem

Let us now take stock and summarize the properties of $\mathcal{M}$ that we have demonstrated so far:

1. *Superconformal symmetry.* The Mellin amplitude $\mathcal{M}$ can be expressed in terms of an auxiliary amplitude $\widetilde{\mathcal{M}}$,

$$\mathcal{M}(s,t;\sigma,\tau) = \widehat{R} \circ \widetilde{\mathcal{M}}(s,t;\sigma,\tau)\,, \tag{4.24}$$

with the help of the difference operator $\widehat{R}$ defined in (4.13).

2. *Bose symmetry.* $\mathcal{M}$ is invariant under permutation of the Mandelstam variables, if the quantum numbers of the external operators are permuted accordingly. For example, when the conformal dimensions of the four half-BPS operators are set to equal $k_i = \mathcal{L}$, Bose symmetry gives the usual crossing relations

$$\begin{aligned} \sigma^{\mathcal{L}}\mathcal{M}(u,t;1/\sigma,\tau/\sigma) &= \mathcal{M}(s,t,;\sigma,\tau)\,, \\ \tau^{\mathcal{L}}\mathcal{M}(t,s;\sigma/\tau,1/\tau) &= \mathcal{M}(s,t;\sigma,\tau)\,. \end{aligned} \tag{4.25}$$

3. *Asymptotics.* The asymptotic behavior of the Mellin amplitude $\mathcal{M}$ is bounded by the flat space scattering amplitude. At large values of the Mandelstam variables, $\mathcal{M}$ should grow linearly

$$\mathcal{M}(\beta s,\beta t;\sigma,\tau) \sim O(\beta) \quad \text{for } \beta \to \infty\,. \tag{4.26}$$

4. *Analytic structure.* $\mathcal{M}$ has only simple poles and there are a finite number of such simple poles in variables s, t, u, located at

$$\begin{aligned} s_0 &= s_M - 2a\,, \quad s_0 \geq 2\,, \\ t_0 &= t_M - 2b\,, \quad t_0 \geq 2\,, \\ u_0 &= u_M - 2c\,, \quad u_0 \geq 2 \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} s_M &= \min\{k_1+k_2, k_3+k_4\} - 2\,, \\ t_M &= \min\{k_1+k_4, k_2+k_3\} - 2\,, \\ u_M &= \min\{k_1+k_3, k_2+k_4\} - 2\,, \end{aligned} \tag{4.28}$$

and a, b, c are non-negative integers. The position of these poles are determined by the twists of the exchanged single-trace operators in the three channels – see Table 2.1 and related discussion in Section 2.1. Moreover, at each simple pole, the residue of the amplitude $\mathcal{M}$ must be a polynomial in the other Mandelstam variable.

These conditions define a very constraining "bootstrap" problem. To start unpacking their content, let us recall that the dependence on the R-symmetry variables σ and τ is polynomial, of degree $\mathcal{L}$ and $\mathcal{L}-2$ for $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, respectively,

$$
\begin{aligned}
\mathcal{M}(s,t;\sigma,\tau) &= \sum_{I+J+K=\mathcal{L}} \sigma^I \tau^J \mathcal{M}_{IJK}(s,t) \ , \\
\widetilde{\mathcal{M}}(s,t;\sigma,\tau) &= \sum_{i+j+k=\mathcal{L}-2} \sigma^i \tau^j \widetilde{\mathcal{M}}_{ijk}(s,t) \ .
\end{aligned}
\tag{4.29}
$$

Bose symmetry amounts to the invariance of $\mathcal{M}_{IJK}(s,t)$ under permutation of (I,J,K) accompanied by simultaneous permutation of (s,t,u), with $u \equiv \sum_{i=1}^4 k_i - s - t$. Analogously, $\widetilde{\mathcal{M}}_{ijk}(s,t)$ is invariant under simultaneous permutation of (i,j,k) and $(s,t,\tilde{u})$, with $\tilde{u} \equiv \sum_{i=1}^4 k_i - s - t - 4$. A little combinatoric argument shows that the number $\mathcal{N}_L$ of independent $\mathcal{M}_{IJK}$ functions is given by

$$
\mathcal{N}_{\mathcal{L}} = \frac{(\mathcal{L}+5)(\mathcal{L}+1)}{12} + \frac{17}{72} + \frac{(-1)^{\mathcal{L}}}{8} + \frac{2}{9}\cos\left(\frac{2\pi\mathcal{L}}{3}\right) . \tag{4.30}
$$

The superconformal Ward identity (4.24) expresses the $\mathcal{N}_{\mathcal{L}}$ functions $\mathcal{M}_{IJK}(s,t)$ in terms of the $\mathcal{N}_{\mathcal{L}-2}$ functions $\widetilde{\mathcal{M}}_{ijk}(s,t)$. Clearly since $\mathcal{N}_{\mathcal{L}} > \mathcal{N}_{\mathcal{L}-2}$ the difference operator $\widehat{R}$ cannot be invertible, *i.e.*, (4.24) represents a non-trivial constraint on $\mathcal{M}$. By assumption 4, $\mathcal{M}_{IJK}(s,t)$ are rational functions of s and t. We will now show that compatibility with (4.24) requires that $\widetilde{\mathcal{M}}_{ijk}(s,t)$ must also be rational functions. (The argument that follows is elementary but slightly elaborate and can be safely skipped on first reading.)

The two sets of R-symmetry monomials $\{\sigma^I\tau^J\}$ and $\{\sigma^i\tau^j\}$ can be conveniently arranged into two equilateral triangles, illustrated respectively by Figure 4.1 and Figure 4.2. The Bose symmetry that relates different R-symmetry monomials corresponds to the S_3 the symmetry of the equilateral triangle. Let us start by considering the monomial 1 in $\mathcal{M}$, which is associated to the coefficient $\mathcal{M}_{0,0,\mathcal{L}}(s,t)$. This monomial can only be reproduced by the monomial 1 in $\widetilde{\mathcal{M}}$, *i.e.*, the term $\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)$, via the action of the

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \tau & & \sigma & & \\
 & \tau^2 & & \tau\sigma & & \sigma^2 & \\
 \iddots & & & & & & \ddots \\
 \tau^{L-2} & & \iddots & \cdots & \ddots & & \sigma^{L-2} \\
 \tau^{L-1} & \tau^{L-2}\sigma & & \cdots & & \tau\sigma^{L-2} & \sigma^{L-1} \\
 \tau^{L} \quad \tau^{L-1}\sigma & \tau^{L-2}\sigma^2 & & \cdots & & \tau^2\sigma^{L-2} & \tau\sigma^{L-1} \quad \sigma^{L}
\end{array}
$$

Figure 4.1: R-symmetry monomials in $\mathcal{M}$.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & \tau & & \sigma & & \\
 & \tau^2 & & \tau\sigma & & \sigma^2 & \\
 \iddots & & & & & & \ddots \\
 \tau^{L-4} & & \iddots & \cdots & \ddots & & \sigma^{L-4} \\
 \tau^{L-3} & \tau^{L-4}\sigma & & \cdots & & \tau\sigma^{L-4} & \sigma^{L-3} \\
 \tau^{L-2} \quad \tau^{L-3}\sigma & \tau^{L-4}\sigma^2 & & \cdots & & \tau^2\sigma^{L-4} & \tau\sigma^{L-3} \quad \sigma^{L-2}
\end{array}
$$

Figure 4.2: R-symmetry monomials in $\widetilde{\mathcal{M}}$.

operator $\widehat{V}$ in $\widehat{R}$,

$$\begin{aligned}\mathcal{M}_{0,0,\mathcal{L}}(s,t) &= \widehat{V} \circ \widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t) \\ &= \widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t-2)\left(\frac{k_2+k_3-t}{2}\right)\left(\frac{k_1+k_4-t}{2}\right) \\ &\quad \times \left(\frac{k_1+k_3-u}{2}\right)\left(\frac{k_2+k_4-u}{2}\right) .\end{aligned} \tag{4.31}$$

We can then $\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)$ in terms of $\mathcal{M}_{0,0,\mathcal{L}}(s,t)$

$$\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t) = \left.\frac{\mathcal{M}_{0,0,\mathcal{L}}(s,t)}{\left(\frac{k_2+k_3-t}{2}\right)\left(\frac{k_1+k_4-t}{2}\right)\left(\frac{k_1+k_3-u}{2}\right)\left(\frac{k_2+k_4-u}{2}\right)}\right|_{t\to t+2} , \tag{4.32}$$

which makes it clear that $\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)$ is rational given that $\mathcal{M}_{0,0,\mathcal{L}}(s,t)$ is assumed to be rational. Similarly, one can easily see that $\sigma^{\mathcal{L}}\mathcal{M}_{\mathcal{L},0,0}(s,t)$ can only be reproduced from $\sigma^{\mathcal{L}-2}\widetilde{\mathcal{M}}_{\mathcal{L}-2,0,0}(s,t)$ via the action of $\sigma^2\widehat{UV}$ and $\tau^{\mathcal{L}}\mathcal{M}_{0,\mathcal{L},0}(s,t)$ can only come from $\tau^{\mathcal{L}-2}\widetilde{\mathcal{M}}_{0,\mathcal{L}-2,0}(s,t)$ with the action $\tau^2\widehat{U}$. These two sets of $\mathcal{M}_{IJK}$ and $\widetilde{\mathcal{M}}_{ijk}$ correspond to the six corners of the two triangles and are in the same orbit under the action of the Bose symmetry. Using the explicit form of the operators $\widehat{UV}$ and $\widehat{U}$ it is apparent that both $\widetilde{\mathcal{M}}_{\mathcal{L}-2,0,0}(s,t)$ and $\widetilde{\mathcal{M}}_{0,\mathcal{L}-2,0}(s,t)$ can be analogously solved and have finitely many poles in the Mandelstam variables. Now let us move on to consider $\sigma\mathcal{M}_{1,0,\mathcal{L}-1}(s,t)$ which receives contribution from $\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)$ with the action of $-\sigma\widehat{V}-\sigma\widehat{UV}+\sigma\widehat{V^2}$ as well as from $\sigma\widetilde{\mathcal{M}}_{1,0,\mathcal{L}-3}(s,t)$ with the action of $\widehat{V}$

$$\mathcal{M}_{1,0,\mathcal{L}-1}(s,t) = (-\widehat{V}-\widehat{UV}+\widehat{V^2})\circ\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)+\widehat{V}\circ\widetilde{\mathcal{M}}_{1,0,\mathcal{L}-3}(s,t) . \tag{4.33}$$

Since we have deduced the finiteness of the number of poles in $\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)$, it is obvious from the above equation that $\widetilde{\mathcal{M}}_{1,0,\mathcal{L}-3}(s,t)$ also has a finite number of poles. By the same logic, one can easily convince oneself that the number of poles in $\widetilde{\mathcal{M}}_{0,1,\mathcal{L}-3}(s,t)$, $\widetilde{\mathcal{M}}_{\mathcal{L}-3,1,0}(s,t)$, $\widetilde{\mathcal{M}}_{\mathcal{L}-3,0,1}(s,t)$, $\widetilde{\mathcal{M}}_{0,\mathcal{L}-3,1}(s,t)$, $\widetilde{\mathcal{M}}_{1,\mathcal{L}-3,0}(s,t)$ is also finite. The strategy is now clear. We start from the corners of the triangle and move along the edges. Each time we encounter a new element of $\widetilde{\mathcal{M}}_{i,j,k}(s,t)$ multiplied by a single difference operator of the type $\widehat{U^mV^n}$ and by recursion we can prove this new term has finitely many poles. After finishing the outer layer of the R-symmetry triangle, we move onto the adjacent layer, again starting from the three corners and then moving along the edges. It is not hard to see that at each step the same situation occurs and we only need to deal with one new element at a time. For example, $\sigma\tau\mathcal{M}_{1,1,\mathcal{L}-2}(s,t)$, which is on the top corner of the second layer, is generated by $\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)$ with the action of $-\sigma\tau\widehat{U}-\sigma\tau\widehat{UV}+\sigma\tau\widehat{U^2}$, $\sigma\widetilde{\mathcal{M}}_{1,0,\mathcal{L}-3}(s,t)$ with $-\tau\widehat{V}-\tau\widehat{U}+\tau\widehat{1}$, $\tau\widetilde{\mathcal{M}}_{0,1,\mathcal{L}-3}(s,t)$ with $-\sigma\widehat{V}-\sigma\widehat{UV}+\sigma\widehat{V^2}$ and $\sigma\tau\widetilde{\mathcal{M}}_{1,1,\mathcal{L}-3}(s,t)$ with $\widehat{V}$. Among these four elements of the auxiliary amplitude $\widetilde{\mathcal{M}}_{0,0,\mathcal{L}-2}(s,t)$, $\sigma\widetilde{\mathcal{M}}_{1,0,\mathcal{L}-3}(s,t)$, $\tau\widetilde{\mathcal{M}}_{0,1,L-3}(s,t)$ belong to the outer layer which are determined to be rational in the previous round. Only the element $\sigma\tau\widetilde{\mathcal{M}}_{1,1,\mathcal{L}-3}(s,t)$ belongs to the inner layer and is acted on by the simple difference operator $\widehat{V}$. This concludes by recursion that $\widetilde{\mathcal{M}}_{1,1,\mathcal{L}-3}(s,t)$ is also rational. In finitely many steps, we can exhaust all the elements of $\widetilde{\mathcal{M}}_{ijk}$. This concludes the proof of rationality of $\widetilde{\mathcal{M}}$. It might at first sight appear that this procedure amounts to an algorithm to invert the difference operator $\widehat{R}$, but of course this is not the case. For general $\mathcal{M}_{IJK}$, one would find contradictory results for some element $\widetilde{\mathcal{M}}_{ijk}$ applying the recursion procedure by following different paths in the triangle.

4.1.3 Contour Subtleties and the Free Correlator

In this subsection we address some subtleties related to s and t integration contours in the Mellin representation. These subtleties are related to the decomposition of the position space correlator into a "free" and a dynamical term. In transforming to Mellin space, we have ignored the term $\mathcal{G}_{\text{free,conn}}$. We are going to see how this term can be recovered by taking the inverse Mellin transform with proper integration contours.

The four-point function calculated from supergravity with the traditional method is a sum of four-point contact diagrams, known as $\bar{D}$-functions. (Their precise definition is given in (2.31)). Through the repeated use of identities obeyed the $\bar{D}$ functions, the supergravity answer can be massaged into a form that agrees with the solution to the superconformal Ward identity – with a singled-out "free" piece. Manipulations of this sort can be found in, *e.g.*, [75, 18, 19, 22]. Most of the requisite identities have an elementary proof either in position space or in Mellin space, but the crucial identity which is key to the separation of the free term, namely

$$\left(\bar{D}_{\Delta_1+1\Delta_2\Delta_3+1\Delta_4} + U\bar{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} + V\bar{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4}\right)\Big|_{\Delta_4=\Delta_1+\Delta_2+\Delta_3} = \prod_{i=1}^{3}\Gamma(\Delta_i), \tag{4.34}$$

requires additional care. The Mellin transform of the rhs is clearly ill-defined. We will now show that the Mellin transform of the lhs is also ill-defined, because while each of the three terms has a perfectly good transform for a finite domain of s and t (known as the "fundamental domain"), the three domains have no common overlap. A suitable regularization procedure is required to make sense of this identity. Let us see this in detail.

Recall that the Mellin transform of an individual $\bar{D}$-function is just a product of Gamma functions. Its fundamental domain can be characterized by the condition that all the arguments of Gamma functions are positive [76]. For the three $\bar{D}$-functions appearing on the lhs of (4.34), we have

$$\begin{aligned}
\bar{D}_{\Delta_1+1\Delta_2\Delta_3+1\Delta_4} &= \tfrac{1}{4}\int_{\mathcal{C}_1} dsdt U^{s/2}V^{t/2}\Gamma[-\tfrac{s}{2}]\Gamma[-\tfrac{t}{2}]\Gamma[\tfrac{s+t+\Delta_1+\Delta_2+\Delta_3-\Delta_4+2}{2}] \\
&\quad\times\Gamma[-\tfrac{s}{2}+\tfrac{\Delta_4+\Delta_3-\Delta_1-\Delta_2}{2}]\Gamma[-\tfrac{t}{2}+\tfrac{\Delta_4+\Delta_1-\Delta_2-\Delta_3}{2}]\Gamma[\tfrac{s+t}{2}+\Delta_2]\,, \\
\bar{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} &= \tfrac{1}{4}\int_{\mathcal{C}_2} dsdt U^{s/2}V^{t/2}\Gamma[-\tfrac{s}{2}]\Gamma[-\tfrac{t}{2}]\Gamma[\tfrac{s+t+\Delta_1+\Delta_2+\Delta_3-\Delta_4+2}{2}] \\
&\quad\times\Gamma[-\tfrac{s}{2}+\tfrac{\Delta_4+\Delta_3-\Delta_1-\Delta_2}{2}-1]\Gamma[-\tfrac{t}{2}+\tfrac{\Delta_4+\Delta_1-\Delta_2-\Delta_3}{2}]\Gamma[\tfrac{s+t}{2}+\Delta_2+1]\,, \\
\bar{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4} &= \tfrac{1}{4}\int_{\mathcal{C}_3} dsdt U^{s/2}V^{t/2}\Gamma[-\tfrac{s}{2}]\Gamma[-\tfrac{t}{2}]\Gamma[\tfrac{s+t+\Delta_1+\Delta_2+\Delta_3-\Delta_4+2}{2}] \\
&\quad\times\Gamma[-\tfrac{s}{2}+\tfrac{\Delta_4+\Delta_3-\Delta_1-\Delta_2}{2}]\Gamma[-\tfrac{t}{2}+\tfrac{\Delta_4+\Delta_1-\Delta_2-\Delta_3}{2}-1]\Gamma[\tfrac{s+t}{2}+\Delta_2+1]\,.
\end{aligned} \tag{4.35}$$

Here

$$\int_{\mathcal{C}_i} dsdt = \int_{s_{0i}-i\infty}^{s_{0i}+i\infty} ds \int_{t_{0i}-i\infty}^{t_{0i}+i\infty} dt\ , \tag{4.36}$$

so the contours are specified by selecting a point inside the fundamental domains, $(s_{0i}, t_{0i}) \in \mathcal{D}_i$. With $\Delta_4 = \Delta_1 + \Delta_2 + \Delta_3$, one finds that the fundamental domains are given by

$$\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \{(s_0, t_0)|\Re(s) < 0, \Re(t) < 0, \Re(s) + \Re(t) > -2\}\ . \tag{4.37}$$

Multiplication by U and V in the second and the third terms, respectively, shifts[3] the domains $\mathcal{D}_2$ and $\mathcal{D}_3$ into new domains $\mathcal{D}'_2$ and $\mathcal{D}'_3$,

$$\begin{aligned} \mathcal{D}'_2 &= \{(s_0,t_0)|\Re(s)<2, \Re(t)<0, \Re(s)+\Re(t)>0\}\ , \\ \mathcal{D}'_3 &= \{(s_0,t_0)|\Re(s)<0, \Re(t)<2, \Re(s)+\Re(t)>0\}\ . \end{aligned} \tag{4.38}$$

This is problematic because

$$\mathcal{D}_1 \bigcap \mathcal{D}'_2 \bigcap \mathcal{D}'_3 = \emptyset\ . \tag{4.39}$$

Clearly it makes no sense to add up the integrands if the contour integrals share no common domain. On the other hand, if one is being cavalier and sums up the integrands anyway, one finds that the total integrand vanishes. This is "almost" the correct result, since the rhs of the identity (4.34) is simply a constant, whose Mellin transform is ill-defined and was indeed set to zero in our analysis in the previous section. We can however do better and reproduce the exact identity if we adopt the following "regularization" prescription: we shift $s+t \to s+t+\epsilon$, with ϵ a small positive real number. After this shift, the three domains develop a small common domain of size ϵ,

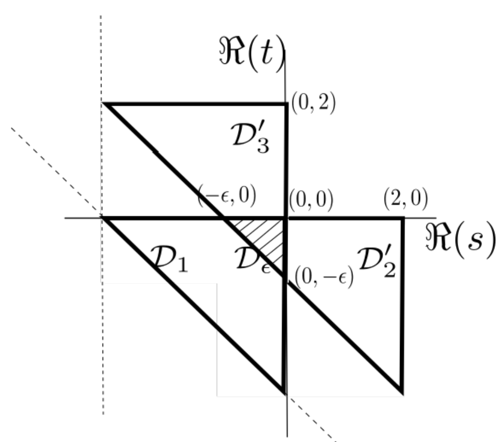

Figure 4.3: The regularized domains. The common domain of size ϵ is depicted as the shaded region.

[3] To absorb $U^m V^n$ outside the integral into $U^{s/2}V^{t/2}$ inside the integral and then shift s and t to bring it back to the form $U^{s/2}V^{t/2}$. Doing so amounts to shift $\mathcal{D}$ to $\mathcal{D}'$ by a vector $(2m,2n)$.

$$\mathcal{D}_1\bigcap\mathcal{D}_2'\bigcap\mathcal{D}_3'\equiv\mathcal{D}_\epsilon=\{(s_0,t_0)|\Re(s)<0,\Re(t)<0,\Re(s)+\Re(t)>-\epsilon\}\ . \tag{4.40}$$

We can therefore place the common integral contour inside $\mathcal{D}_\epsilon$ and combine the integrands,

$$\begin{aligned}\text{LHS}=&\frac{1}{4}\int_{\mathcal{D}_\epsilon}dsdtU^{s/2}V^{t/2}(\frac{s+t+\epsilon}{2}-\frac{s}{2}-\frac{t}{2})\Gamma[-\frac{s}{2}]\Gamma[-\frac{t}{2}]\Gamma[\frac{s+t+\epsilon}{2}]\\&\times\Gamma[-\frac{s}{2}+\Delta_3]\Gamma[-\frac{t}{2}+\Delta_1]\Gamma[\frac{s+t}{2}+\Delta_2]\\=&\frac{1}{4}\int_{\mathcal{D}_\epsilon}dsdtU^{s/2}V^{t/2}\frac{\epsilon}{2}\Gamma[-\frac{s}{2}]\Gamma[-\frac{t}{2}]\Gamma[\frac{s+t+\epsilon}{2}]\\&\times\Gamma[-\frac{s}{2}+\Delta_3]\Gamma[-\frac{t}{2}+\Delta_1]\Gamma[\frac{s+t}{2}+\Delta_2]\ .\end{aligned} \tag{4.41}$$

As $\epsilon\to0$, we can just substitute $s=t=0$ into the non-singular part of the integrand. The resulting integral is easily evaluated,

$$\begin{aligned}\text{LHS}=&\frac{1}{2}\Gamma[\Delta_1]\Gamma[\Delta_2]\Gamma[\Delta_3]\int_{\mathcal{D}_\epsilon}\frac{ds}{2}\frac{dt}{2}\epsilon\Gamma[-\frac{s}{2}]\Gamma[-\frac{t}{2}]\Gamma[\frac{s+t+\epsilon}{2}]=\Gamma[\Delta_1]\Gamma[\Delta_2]\Gamma[\Delta_3]\\=&\,\text{RHS}\,.\end{aligned} \tag{4.42}$$

This amounts to a "proof" of the identity (4.34) directly in Mellin space. This exercise contains a useful general lesson. As we have already remarked, the identity (4.34) is responsible for generating the term $\mathcal{G}_{\text{free,conn}}$ by collapsing sums of $\bar{D}$ functions in the supergravity answer. We have shown that it is consistent to treat the Mellin transform of $\mathcal{G}_{\text{free,conn}}$ as "zero", provided that we are careful about the s, t integration contours in the inverse Mellin transform. In general, when one is adding up integrands, one should make sure the integrals share the same contour, which may require a regularization procedure of the kind we have just used. A naively "zero" Mellin amplitude can then give nonzero contributions to the integral if the contour is pinched to an infinitesimal domain where the integrand has a pole. In Appendix C we illustrate in the simplest case of equal weights $k_i=2$ how the free field correlator is correctly reproduced by this mechanism.

We conclude by alerting the reader about another small subtlety. The free term $\mathcal{G}_{\text{free,conn}}$ depends on the precise identification of the operators dual to the supergravity modes s_p. As explained in footnote 1, if one adopts the scheme where the fields s_k contain no derivative cubic couplings, the dual operators are necessarily admixtures of single- and multi-trace operators. While the multi-trace pieces are in general subleading, they can affect the free-field four-point function if the four weights are sufficiently "unbalanced". This phenomenon was encountered in [21, 22], where the four-point functions with weights $(2, 2, k, k)$ were evaluated from supergravity. A discrepancy was found for $k \geq 4$ between the function $\mathcal{G}_{\text{free,conn}}$ obtained by writing the supergravity result in the split form (4.3) and the free-field result obtained in free field theory from Wick contractions, assuming that the operators are pure single-traces. The resolution is that supergravity is really computing the four-point function of more complicated operators with multi-trace admixtures. Note that the contribution to the four-point functions from the multi-trace terms takes the form of a product of two- and three-point functions of one-half BPS operators, and is thus protected [34]. The ambiguity in the precise identification of the dual operators can then only affect $\mathcal{G}_{\text{free,conn}}$ and not the dynamical part.

4.2 General Solution for $AdS_5 \times S^5$

Experimentation with low-weight examples led us to the following ansatz for $\widetilde{\mathcal{M}}$,

$$\widetilde{\mathcal{M}}(s,t,\tilde{u};\sigma,\tau) = \sum_{\substack{i+j+k=\mathcal{L}-2,\\ 0\leq i,j,k\leq \mathcal{L}-2}} \frac{a_{ijk}\sigma^i\tau^j}{(s-s_M+2k)(t-t_M+2j)(\tilde{u}-\tilde{u}_M+2i)} \tag{4.43}$$

where

$$\begin{aligned} s_M &= \min\{k_1+k_2, k_3+k_4\} - 2\ , \\ t_M &= \min\{k_1+k_4, k_2+k_3\} - 2\ , \\ \tilde{u}_M &= \min\{k_1+k_3, k_2+k_4\} - 2\ . \end{aligned} \tag{4.44}$$

The reader can check that this ansatz leads to an $\mathcal{M}$ that satisfies the asymptotic requirement, obeys Bose symmetry and has simple poles at the required location. The further requirements that the poles have polynomials residues fixes the coefficients a_{ijk} uniquely up to normalization,

$$a_{ijk} = \frac{C_{k_1k_2k_3k_4}\binom{\mathcal{L}-2}{i,j,k}}{(1+\frac{|k_1-k_2+k_3-k_4|}{2})_i(1+\frac{|k_1+k_4-k_2-k_3|}{2})_j(1+\frac{|k_1+k_2-k_3-k_4|}{2})_k}\ , \tag{4.45}$$

where $\binom{\mathcal{L}-2}{i,j,k}$ is the trinomial coefficient. The overall normalization

$$C_{k_1k_2k_3k_4} = \frac{f(k_1,k_2,k_3,k_4)}{N^2} \tag{4.46}$$

cannot be fixed from our homogenous consistency conditions.

There are several ways to fix this normalization. In principle, it can be determined by transforming back to the position-space expression (4.3). As we shall show below, the term $\mathcal{G}_{\text{free,conn}}$ arises as a regularization effect in the inverse Mellin transformation. The constant $f(k_1,k_2,k_3,k_4)$ is fixed by requiring that the regularization procedure gives the correctly normalized free-field correlator. In practice, this is very cumbersome, and it is easier to take instead $\mathcal{G}_{\text{free,conn}}$ as an input from free-field theory. The overall normalization of $\mathcal{M}$ is then fixed by imposing the cancellation of spurious singularity associated to single-trace long operators [23], which are separately present in $\mathcal{G}_{\text{free,conn}}$ and in $R\,\mathcal{H}$ but must cancel in the sum. This method has been used in [77] to determine $f(p,p,q,q)$, the normalization in all cases with pairwise equal weights. A more elegant method appeared recently in [78] where the normalization for the most general correlator is fixed. This method involves taking a light-like limit where the points x_1, x_2, x_3, x_4 are sequentially light-like separated. It exploits a property of the correlator that

$$\lim_{U,V\to 0} \mathcal{P}^{-1} G_{\text{conn}}(x_i,t_i) = 0\ , \quad \frac{U}{V} \text{ fixed}\ , \tag{4.47}$$

which can be proved on general grounds [78]. And $\mathcal{P}$ is a product of two-point functions

$$\mathcal{P} = \left(\frac{t_{34}}{x_{34}^2}\right)^{\frac{k_3+k_4+k_2-k_1}{2}} \left(\frac{t_{14}}{x_{14}^2}\right)^{\frac{k_1+k_4-k_2-k_3}{2}} \left(\frac{t_{13}}{x_{13}^2}\right)^{\frac{k_1+k_3-k_2-k_4}{2}} \left(\frac{t_{12}}{x_{12}^2}\right)^{k_2} . \tag{4.48}$$

Using this vanishing relation, the remaining factor is fixed to be

$$f(k_1,k_2,k_3,k_4) = \frac{-2^5 k_1k_2k_3k_4}{\left(\frac{|k_1+k_2-k_3-k_4|}{2}\right)!\left(\frac{|k_1+k_4-k_2-k_3|}{2}\right)!\left(\frac{|k_2+k_4-k_1-k_3|}{2}\right)!(\mathcal{L}-2)!}\ . \tag{4.49}$$

Our result of one-half BPS four-point function with general weights has recently led to impressive progress in the quantitative understanding of the $\mathcal{N}=4$ SYM at large N and infinite 't Hooft coupling, where a great deal of its OPE data has been learned. For example, as we discussed in Section

3.1.1, the double poles in the reduced Mellin amplitude give rise to, after the s integration, $\log U$ terms which encode tree-level anomalous dimensions of double-trace operators. The complete spectrum of the double-trace operators has been recently computed in [78] to the order $O(1/N^2)$.[4]. In obtaining the spectrum, a complexity which one faces is the operator mixing of double-trace operators. To solve this mixing problem, our general formula is a necessary input. Moreover, inputting the tree level data, one is able to bootstrap loop corrections to IIB supergravity on $AdS_5 \times S^5$, see [80, 79, 77, 81, 82] for recent progress using various techniques. It would be interesting to repeat the loop level analysis in Mellin space.

4.2.1 Uniqueness for $k_i = 2$

Uniqueness of the ansatz (4.43) is in general difficult to prove. However in simple examples it is possible to solve the algebraic problem directly, thereby proving that the answer is unique. In this subsection we demonstrate it for the simplest case, the equal-weights case with $k = 2$. This case is particularly simple because $\widetilde{\mathcal{M}}$ has no σ, τ dependence.

Recall that the Mellin amplitude $\mathcal{M}$ has simple poles in s, t and u whose positions are restricted by the condition (4.28). Specifically in the case of $k_i = 2$, it means that the Mellin amplitude can only have simple poles at $s = 2$, $t = 2$ and $u = 2$. On the other hand, $\widetilde{\mathcal{M}}$ must also have poles because the Pochhammer symbols in the difference operators (4.14) do not introduce additional poles. To fix the position of these poles in $\widetilde{\mathcal{M}}$, let us look at the R-symmetry monomial $\sigma^I\tau^J$ in $\mathcal{M}(s,t;\sigma,\tau)$ with $I = J = 0$. The $\sigma^I\tau^J$ term in $\mathcal{M}(s,t;\sigma,\tau)$ with $I = J = 0$ can then only be produced from $\widetilde{\mathcal{M}}(s,t)$ with the action of the term $\widehat{V}$ in (4.13)

$$\widehat{V} \circ \widetilde{\mathcal{M}}(s,t) = \widetilde{M}(s,t-2)\left[\left(\frac{4-t}{2}\right)_1\left(\frac{4-u}{2}\right)_1\right]^2 . \tag{4.50}$$

For s to have simple pole at $s = 2$ in $\mathcal{M}$, it is easy to see that the only possible s-pole in $\widetilde{\mathcal{M}}(s,t)$ is a simple pole at $s = 2$. For t, a simple pole at $t = 0$ in $\widetilde{\mathcal{M}}(s,t)$ is allowed, which after the shift on the right side of (4.50) gives a simple pole at $t = 2$ in $\mathcal{M}$. But there is also an additional pole in t allowed due to the presence of the Pochhammer symbol. Since the

[4]Previous partial results have been reported in [79, 77].

Pochhammer symbol gives a degree-two zero at $t = 4$ we can have a pole at $t = 2$ in $\widetilde{\mathcal{M}}(s,t)$ with pole degree up to two. These two possibilities exhaust all the allowed t-poles in $\widetilde{\mathcal{M}}(s,t)$ that are compatible with the pole structure of $\mathcal{M}$. Now the story for $\tilde{u}$-poles is exactly the same as t. To see this, we note that under the shift $t \to t - 2$,

$$\tilde{u} \to \tilde{u} + 2 = (u - 4) + 2 = u - 2 \ . \tag{4.51}$$

By the same argument $\tilde{u}$ can have in $\widetilde{\mathcal{M}}(s,t)$ a simple pole at $\tilde{u} = 0$ and at most a double pole at $\tilde{u} = 2$.

Now we use the constraints from Bose symmetry (actually crossing symmetry in this case) and the asymptotic condition to further narrow down the possibilities. Bose symmetry requires

$$\widetilde{\mathcal{M}}(s,t) = \widetilde{\mathcal{M}}(s,\tilde{u}) = \widetilde{\mathcal{M}}(t,s) \ . \tag{4.52}$$

Since $\widetilde{\mathcal{M}}(s,t)$ cannot have a pole at $s = 0$, the poles at $t = 0$, $\tilde{u} = 0$ are prohibited. On the other hand the asymptotic condition further requires $\mathcal{M}(s,t)$ to have growth rate one at large s, t, u. Consequently by simple power counting $\widetilde{\mathcal{M}}(s,t)$ should have growth rate -3. This leaves us with the unique crossing symmetric ansatz

$$\widetilde{\mathcal{M}}(s,t) \propto \frac{1}{(s-2)(t-2)(\tilde{u}-2)} \tag{4.53}$$

which is just our solution (4.43).

4.3 Formulating an Algebraic Bootstrap Problem: $AdS_7 \times S^4$

4.3.1 Rewriting the Superconformal Ward Identity

Solution in Position Space

We now turn to set up the algebraic bootstrap problem for $AdS_7 \times S^4$. The theory has a superconformal group $OSp(8^*|4)$, whose bosonic subgroup is the direct product of the conformal group $SO(6,2)$ and R-symmetry group $SO(5)$. We follow the same strategy for $AdS_5 \times S^5$ and first solve the superconformal Ward identity (2.23) in position with $\epsilon = 2$. For the simplicity of the discussion, we will focus on the case where external operators all have weight k.

We can first obtain a partial solution to the superconformal Ward identity by restricting the four-point function to a special slice of R-symmetry cross ratios such that $\alpha = \alpha' = 1/\chi'$. Then the superconformal Ward identity (2.23) reduces to

$$\chi' \partial_{\chi'} \mathcal{G}_k(\chi, \chi'; 1/\chi', 1/\chi') = 0 \ , \tag{4.54}$$

whose solution is simply any "holomorphic" function[5] of χ,

$$\mathcal{G}_k(\chi, \chi'; 1/\chi', 1/\chi') = f(\chi) \ . \tag{4.55}$$

Up to kinematic factors, the function $f(\chi)$ coincides with the four-point correlator of the two-dimensional chiral algebra associated to the $(2,0)$ theory by the cohomological procedure introduced in [83, 84]. There is a compelling conjecture [84] that the chiral algebra associated to the $(2,0)$ theory of type A_n is the familiar $\mathcal{W}_n$ algebra, with central charge $c_{2d} = 4n^3 - 3n - 1$. In our holographic setting, we are instructed to take a suitable large n limit of the $\mathcal{W}_n$ algebra, as explained in detail in [84]. In that limit, the structure constants of the $\mathcal{W}_n$ algebra were matched with the three-point functions of the one-half BPS operators computed holographically by standard supergravity methods. In this thesis, we have used the position space method to compute holographic *four*-point functions of one-half BPS operators. As an important consistency check, we will match their "holomorphic" piece $f(\chi)$ with the corresponding four-point functions in the $\mathcal{W}_{n\to\infty}$ algebra.

The full solution of the superconformal Ward identity (2.23) was found in [37]. We reproduce it here with a few crucial typos fixed. A general solution $\mathcal{G}_k$ of (2.23) can be written as

$$\mathcal{G}_k(U, V; \sigma, \tau) = \mathcal{F}_k(U, V; \sigma, \tau) + \mathcal{K}_k(U, V; \sigma, \tau) \ , \tag{4.56}$$

where $\mathcal{F}_k$ and $\mathcal{K}_k$ are respectively an "inhomogeneous" solution" and a "homogenous" solution. By this we mean that upon performing the "twist" $\alpha = \alpha' = 1/\chi'$, $\mathcal{F}_k$ becomes a purely "holomorphic" function of χ, while $\mathcal{K}_k$ must vanish identically. The homogenous part $\mathcal{K}_k$ can further be expressed

[5] In Euclidean signature, the variables χ and χ' are complex conjugate of each other, so this terminology is appropriate. In Lorentzian signature χ and χ' are instead real independent variables. Hence "holomorphic" in quotes.

in terms of a differential operator Υ acting on an unconstrained function $\mathcal{H}(U,V;\sigma,\tau)$, which is a polynomial in σ and τ of degree $k-2$. Explicitly,[6]

$$\begin{aligned}\mathcal{K}_k(U,V;\sigma,\tau) = &\big(\sigma^2\mathcal{D}'_\epsilon UV + \tau^2\mathcal{D}'_\epsilon U + \mathcal{D}'_\epsilon V - \sigma\mathcal{D}'_\epsilon V(U+1-V) \\ &-\tau\mathcal{D}'_\epsilon(U+V-1) - \sigma\tau\mathcal{D}'_\epsilon U(V+1-U)\big)\mathcal{H}_k(U,V;\sigma,\tau) \\ &=:\Upsilon\circ\mathcal{H}_k(U,V;\sigma,\tau)\ ,\end{aligned} \tag{4.57}$$

where the differential operator $\mathcal{D}'_\epsilon$ is defined as

$$\begin{aligned}\mathcal{D}'_\epsilon &:= \Big[D_\epsilon - \frac{\epsilon}{V}(D_0^+ - D_1^+ + \epsilon\partial_\sigma\sigma)\tau\partial_\tau \\ &\qquad + \frac{\epsilon}{UV}(-VD_1^+ + \epsilon(V\partial_\sigma\sigma + \partial_\tau\tau - 1))(\partial_\sigma\sigma + \partial_\tau\tau)\Big]^{\epsilon-1}\ , \\ D_\epsilon &:= \frac{\partial^2}{\partial\chi\partial\chi'} - \epsilon\frac{1}{\chi-\chi'}(\frac{\partial}{\partial\chi} - \frac{\partial}{\partial\chi'})\ , \\ D_0^+ &:= \frac{\partial}{\partial\chi} + \frac{\partial}{\partial\chi'}\ , \\ D_1^+ &:= \chi\frac{\partial}{\partial\chi} + \chi'\frac{\partial}{\partial\chi'}\ .\end{aligned} \tag{4.58}$$

While the expression of the differential operator Υ is not very transparent, its transformation properties under crossing however are surprisingly simple. Let g_1, g_2 be the two generators of the crossing-symmetry group S_3 under which the cross ratios transform as

$$\begin{aligned} g_1: &\qquad U\to\frac{U}{V}\ ,\ \ V\to\frac{1}{V}\ ,\ \ \sigma\to\tau\ ,\ \ \tau\to\sigma\ , \\ g_2: &\qquad U\to\frac{1}{U}\ ,\ \ V\to\frac{V}{U}\ ,\ \ \sigma\to\frac{1}{\sigma}\ ,\ \ \tau\to\frac{\tau}{\sigma}\ .\end{aligned} \tag{4.59}$$

We have found that Υ satisfies[7]

$$\begin{aligned} g_1\circ\Upsilon &= \sigma^{\frac{1}{2}(-\gamma-\rho)}\tau^{\rho/2}U^{-\gamma}V^{-\rho}\Upsilon\sigma^{\frac{\gamma+\rho}{2}}\tau^{-\frac{\rho}{2}}U^{\gamma}V^{\rho}\ , \\ g_2\circ\Upsilon &= \sigma^{\rho-\frac{\gamma}{2}}\tau^{-\rho}V^{2\rho}U^{-\gamma}\Upsilon\sigma^{\frac{\gamma}{2}-\rho-2}\tau^{\rho}V^{-2\rho}U^{\gamma}\ ,\end{aligned} \tag{4.60}$$

[6]Here and below, the parameter ϵ takes the fixed value 2. We keep it as ϵ to facilitate compare with the expressions in [37], but we stress that the solution to the superconformal Ward identity takes this particular form only for $d=6$.

[7]It is understood here that both sides of (4.60) are acting on the same arbitrary function of the cross-ratios.

where γ and ρ are arbitrary parameters.

We can always find a decomposition of $\mathcal{G}_k$ such that the two functions $\mathcal{F}_k$ and $\mathcal{H}_k$ do not mix into each other under crossing. Then the full correlator $\mathcal{G}_k$ and and the inhomogenous part $\mathcal{F}_k$ have the same crossing properties

$$\mathcal{G}_k(U,V;\sigma,\tau) = \left(\frac{U^2\tau}{V^2}\right)^k \mathcal{G}_k(V,U;\sigma/\tau,1/\tau) = \left(U^2\sigma\right)^k \mathcal{G}_k(1/U,V/U;1/\sigma,\tau/\sigma)\,,$$
$$\mathcal{F}_k(U,V;\sigma,\tau) = \left(\frac{U^2\tau}{V^2}\right)^k \mathcal{F}_k(V,U;\sigma/\tau,1/\tau) = \left(U^2\sigma\right)^k \mathcal{F}_k(1/U,V/U;1/\sigma,\tau/\sigma)\,. \tag{4.61}$$

Using the crossing identities obeyed by the operator Υ, it is then easy to find the crossing relations obeyed by the unconstrained function $\mathcal{H}_k$,

$$\mathcal{H}_k(U,V;\sigma,\tau) = \mathcal{H}_k(U/V,1/V;\tau,\sigma) = U^{2k}\sigma^{k-2}\mathcal{H}_k(1/U,V/U;1/\sigma,\tau/\sigma)\,. \tag{4.62}$$

In closing, we should emphasize that the decomposition (4.56) is not unique, since obviously one can add any "homogeneous" term to $\mathcal{F}_k$ and subtract the same term from $\mathcal{K}_k$. In the case of $\mathcal{N}=4$ super Yang-Mills, where the solution of the superconformal Ward identity takes a similar form, there is a natural choice for $\mathcal{F}_k$, namely the value of the correlator in the free field limit: $\mathcal{F}_k$ is then a simple rational function of U and V. *A priori* there is no reason that an analogous natural choice for $\mathcal{F}_k$ should exist in the $(2,0)$ theory, but we will find experimentally that there is one, even in the absence (obvious) connection with free field theory.

Solution in Mellin Space

In position space, superconformal symmetry of the four-point function is encoded in the solution of the superconformal Ward identity,[8]

$$\mathcal{G}_{k,\mathrm{conn}} = \mathcal{F}_{k,\mathrm{conn}} + \Upsilon\circ\mathcal{H}_{k,\mathrm{conn}}\,. \tag{4.63}$$

Now we begin to translate this solution into Mellin space.

We start by writing the Mellin transformation of the left-hand side of (4.63),

$$\mathcal{G}_{k,\mathrm{conn}} = \int_{-i\infty}^{i\infty}\frac{ds}{2}\frac{dt}{2}U^{\frac{s}{2}}V^{\frac{t}{2}-2k}\mathcal{M}_k(s,t;\sigma,\tau)\Gamma^2[-\frac{s}{2}+2k]\Gamma^2[-\frac{t}{2}+2k]\Gamma^2[-\frac{u}{2}+2k]\,. \tag{4.64}$$

[8] As the Mellin transformation of the disconnected part of the correlator is ill-defined, we focus on the connected part.

Consider now the right-hand side of (4.63). The Mellin transform of the rational function $\mathcal{F}_{k,\text{conn}}$ is ill-defined. As explained in [4], it can be defined as "zero".[9] On the other hand, we can write a Mellin representation for the dynamical function $\mathcal{H}_k$ in terms of an "auxiliary" Mellin amplitude $\widetilde{\mathcal{M}}_k(s,t;\sigma,\tau)$

$$\mathcal{H}_k = \int_{-i\infty}^{i\infty} \frac{ds}{2}\frac{dt}{2} U^{\frac{s}{2}+1} V^{\frac{t}{2}-2k+1} \widetilde{\mathcal{M}}_k(s,t;\sigma,\tau)\Gamma^2[-\frac{s}{2}+2k]\Gamma^2[-\frac{t}{2}+2k]\Gamma^2[-\frac{\tilde{u}}{2}+2k] \tag{4.65}$$

where $\tilde{u} = u - 6$. As we demonstrate the shift has the virtue of giving simple transformation properties to $\widetilde{\mathcal{M}}_k(s,t;\sigma,\tau)$ under crossing,

$$\begin{aligned} \sigma^{p-2}\widetilde{\mathcal{M}}_k(\tilde{u},t;1/\sigma,\tau/\sigma) &= \widetilde{\mathcal{M}}_k(s,t,;\sigma,\tau) \ , \\ \tau^{p-2}\widetilde{\mathcal{M}}_k(t,s;\sigma/\tau,1/\tau) &= \widetilde{\mathcal{M}}_k(s,t;\sigma,\tau) \ . \end{aligned}$$

In the auxiliary amplitude $\widetilde{\mathcal{M}}_k(s,t;\sigma,\tau)$, the triplet variables $(s,t,\tilde{u})$ replaces (s,t,u) to become the set of variables that permute under crossing. This becomes especially evident after we restore all the factors of x_{ij}^2 and t_{ij}. Restoring this factor will also facilitate the extraction of the difference operator. Let us see this in detail.

We first note that the combination

$$a^k A^{-2k} \Upsilon \circ \mathcal{H}_k \tag{4.66}$$

is crossing invariant, since it has the same crossing properties as G_k. Upon inserting the inverse Mellin transformation (4.65) into (4.66) and decomposing the auxiliary amplitude with respect to the R-symmetry monomials,

$$\widetilde{\mathcal{M}}(s,t;\sigma,\tau) = \sum_{l+m+n=k-2} \sigma^m \tau^n \ \widetilde{\mathcal{M}}_{k,\,lmn}(s,t) \ , \tag{4.67}$$

we find

$$(\frac{a}{A^2})^k \ \Upsilon \circ \sum_{l+m+n=k-2} \int_{-i\infty}^{i\infty} \frac{ds}{2} \ \frac{dt}{2} \ U^{s/2+1} V^{t/2-2k+1} \sigma^m \tau^n \widetilde{\mathcal{M}}_{k,\,lmn}(s,t) \ \tilde{\Gamma}_3(s,t) \ . \tag{4.68}$$

[9] $\mathcal{F}_{k,\text{conn}}$ can be recovered as a subtle regularization effect in properly defining the contour integrals of the inverse Mellin transformation [4].

Here the factor $\tilde{\Gamma}_3(s,t)$ is short-hand for $\Gamma^2[-\frac{s}{2}+2k]\Gamma^2[-\frac{t}{2}+2k]\Gamma^2[-\frac{\tilde{u}}{2}+2k]$. We now let the differential operator Υ act on the monomial $U^{s/2+1}V^{t/2-2k+1}\sigma^m\tau^n$, leading to the following integral

$$\int_{-i\infty}^{i\infty}\frac{ds}{2}\,\frac{dt}{2}\sum_{l+m+n=k-2}a^lb^mc^nA^{s/2-2k}B^{\tilde{u}/2-2k}C^{t/2-2k}\;\Theta\;\widetilde{\mathcal{M}}_{k,\,lmn}(s,t)\;\tilde{\Gamma}_3(s,t)\;, \tag{4.69}$$

where $U=A/B$, $V=C/B$, $\sigma=b/a$, $\tau=c/a$ have been substituted back into the expression. The factor Θ is a polynomial of A, B, C, a, b, c and s, t obtained from the action of the differential operator Υ on the monomial $U^{s/2+1}V^{t/2-2k+1}\sigma^m\tau^n$. To write an explicit expression for Θ, it is useful further to introduce the following combinations of Mandelstam variables,

$$\begin{aligned}X&=s+4l-4k+2\;,\\Y&=t+4n-4k+2\;,\\\tilde{Z}&=\tilde{u}+4m-4k+2\;.\end{aligned} \tag{4.70}$$

In terms of these variables, Θ reads

$$\begin{aligned}\Theta=&-\frac{1}{4}\Big[abC^3X\tilde{Z}+aB^3cXY+A^3bcY\tilde{Z}\\&+A^2bC(c(X+4)\tilde{Z}-(Y+2)\tilde{Z}(a-b+c))+AbC^2(a(Y+4)\tilde{Z}-(X+2)\tilde{Z}(a-b+c))\\&+aBC^2(X(\tilde{Z}+2)(a-b-c)+bX(Y+4))+A^2Bc(b(4+X)Y-(a+b-c)Y(2+\tilde{Z}))\\&+aB^2C((a-b-c)X(2+Y)+cX(4+\tilde{Z}))\\&+AB^2c((-a-b+c)(2+X)Y+aY(4+\tilde{Z}))\\&+ABC\big(a^2(Y+2)(\tilde{Z}+2)+b^2(X+2)(Y+2)+c^2(X+2)(\tilde{Z}+2)\\&\quad+bc(2+X)(4+X)+ab(2+Y)(4+Y)+ac(2+\tilde{Z})(4+\tilde{Z})\big)\Big]\;.\end{aligned} \tag{4.71}$$

The reader should not focus on this complicated expression because further manipulations will soon lead to a major simplification. At this stage we only want to point out that the above expression of Θ can be checked to be crossing invariant under any permutation of the triplets

$$(a,A,s)\;,\qquad(b,B,\tilde{u})\;,\qquad(c,C,t)\;. \tag{4.72}$$

Crossing invariance of (4.69) implies the following crossing identities for

$\widetilde{\mathcal{M}}_{k,\,lmn}(s,t)$,

$$\begin{aligned} \widetilde{\mathcal{M}}_{k,\,nml}(t,s) &= \widetilde{\mathcal{M}}_{k,\,lmn}(s,t)\ , \\ \widetilde{\mathcal{M}}_{k,\,mnl}(\tilde{u},t) &= \widetilde{\mathcal{M}}_{k,\,lmn}(s,t)\ , \end{aligned} \tag{4.73}$$

from which the crossing identities (4.66) of $\widetilde{\mathcal{M}}_k(s,t;\sigma,\tau)$ immediately follow.

As in Section 4.1, we should reinterpret the monomials of A, B, C in (4.71) as difference operators acting on functions of s, t in the integrand, thus promoting the factor Θ to an operator $\widehat{\Theta}$. This operator $\widehat{\Theta}$ can be written in a compact form if the respective shift on X, Y and $\tilde{Z}$ has first been performed, as we now show. All monomials that appear in Θ have the form $A^\alpha B^{3-\alpha-\beta}C^\beta$. Multiplying an inverse Mellin integral by such a monomial, we have

$$\begin{aligned} &A^\alpha B^{3-\alpha-\beta}C^\beta \int_{\mathcal{C}} dsdt\, A^{s/2-2k}B^{\tilde{u}/2-2k}C^{t/2-2k}F(s,t) = \\ &\int_{\mathcal{C}'} dsdt A^{s/2-2k}B^{u/2-2k}C^{t/2-2k}F(s-2\alpha,t-2\beta)\ . \end{aligned} \tag{4.74}$$

(The shift of the integration contour is important in producing rational terms by the mechanism discussed in Section 4.1.3 and Appendix C. Here we are focusing on the Mellin amplitude and ignore contour issues.) Note that in the first term we use the shifted Mandelstam variable $\tilde{u} = u - 6 = 8k - s - t - 6$, while the unshifted u appears in the second term. We conclude that multiplication by the monomial $A^\alpha B^{3-\alpha-\beta}C^\beta$ corresponds in Mellin space to a difference operator that shifts $s \to s - 2\alpha$ and $t \to t - 2\beta$.

Interpreting every monomials in Θ in this fashion we find a difference operator $\widehat{\Theta}$. We can make the expression of $\widehat{\Theta}$ very compact by performing the shift in two stages: first we shift on the factor of X, Y, $\tilde{Z}$ multiplying each monomial $A^\alpha B^{3-\alpha-\beta}C^\beta$ and bring it the left; then $A^\alpha B^{3-\alpha-\beta}C^\beta$ remains an operator to act on whatever is in the integrand on the right. We arrive at the following simple expression,

$$\widehat{\Theta} = -\frac{1}{4}\Big((XY)\widehat{B\mathfrak{R}} + (XZ)\widehat{C\mathfrak{R}} + (YZ)\widehat{A\mathfrak{R}}\Big) \tag{4.75}$$

where we defined an "unshifted" Z variable

$$Z := \tilde{Z} + 6 = u + 4m - 4k + 2 \tag{4.76}$$

and the crossing-invariant factor $\mathfrak{R}$ is given by[10]

$$\mathfrak{R} = a^2BC+b^2AC+c^2AB+abC(-A-B+C)+acB(-A+B-C)+bcA(A-B-C) \tag{4.77}$$

The expressions $\widehat{A\mathfrak{R}}$, $\widehat{B\mathfrak{R}}$, $\widehat{C\mathfrak{R}}$ are written shorthands that should be understood as follows: one first expands $A\mathfrak{R}$, $B\mathfrak{R}$, $C\mathfrak{R}$ into monomials $A^\alpha B^{3-\alpha-\beta}C^\beta$ and then regards each of them as the operator $\widehat{A^\alpha B^{3-\alpha-\beta}C^\beta}$. These operators will only act on objects multiplied from the right and will no longer shift the X, Y, Z factors multiplied from the left.

We can now give the explicit action of $\widehat{A^\alpha B^{3-\alpha-\beta}C^\beta}$ as an operator that transforms a term of $\widetilde{\mathcal{M}}_k$ into a term of $\mathcal{M}_k$,

$$\begin{aligned} \widehat{A^\alpha B^{3-\alpha-\beta}C^\beta} \circ \widetilde{\mathcal{M}}_{k,\, lmn}(s,t) :&= \widetilde{\mathcal{M}}_{k,\, lmn}(s-2\alpha, t-2\beta) \\ &\times \frac{\Gamma^2[-\frac{s}{2}+2k+\alpha]\Gamma^2[-\frac{t}{2}+2k+\beta]\Gamma^2[-\frac{u}{2}+2k+(3-\alpha-\beta)]}{\Gamma^2[-\frac{s}{2}+2k]\Gamma^2[-\frac{t}{2}+2k]\Gamma^2[-\frac{u}{2}+2k]} \ . \end{aligned} \tag{4.78}$$

This action is obtained by applying the aforementioned shift of s and t on the integrand and taking into consideration the difference of Gamma function factors between the definitions (4.64) and (4.69).

All in all, the superconformal Ward identity implies that the full Mellin amplitude $\mathcal{M}_k$ can be written in terms of an auxiliary amplitude $\widetilde{\mathcal{M}}_k$ acted upon by the difference operator $\widehat{\Theta}$,

$$\mathcal{M}_k = \widehat{\Theta} \circ \widetilde{\mathcal{M}}_k \ . \tag{4.79}$$

The operator $\widehat{\Theta}$ is given by (4.75) where each monomial operator acts as in (4.81).

Finally, let us mention that the generalization of the difference operator to amplitudes with unequal weights is straightforward. Following a similar procedure, we find that in the general case we only need to modify the definitions of X, Y, Z by

$$\begin{aligned} X &= s + 4l + 2 - 2\min\{k_1+k_2, k_3+k_4\} \ , \\ Y &= t + 4n + 2 - 2\min\{k_1+k_4, k_2+k_3\} \ , \\ Z &= u + 4m + 2 - 2\min\{k_1+k_3, k_2+k_4\} \ , \end{aligned} \tag{4.80}$$

[10] Curiously, this factor also appeared in the solution of the $4d$ $\mathcal{N}=4$ superconformal Ward identity (4.20). We don't have a deep understanding of this observation.

and let each monomial act as

$$
\begin{aligned}
A^\alpha \widehat{B^{3-\alpha-\beta}} C^\beta \circ \widetilde{\mathcal{M}}_{lmn}(s,t) \equiv \widetilde{\mathcal{M}}_{lmn}(s-2\alpha, t-2\beta) \times \left(\frac{2(k_1+k_2)-s}{2}\right)_\alpha \\
\times \left(\frac{2(k_3+k_4)-s}{2}\right)_\alpha \left(\frac{2(k_1+k_4)-t}{2}\right)_\beta \left(\frac{2(k_2+k_3)-t}{2}\right)_\beta \\
\times \left(\frac{2(k_1+k_3)-u}{2}\right)_{3-\alpha-\beta} \left(\frac{2(k_2+k_4)-u}{2}\right)_{3-\alpha-\beta} .
\end{aligned}
\tag{4.81}
$$

4.3.2 An Algebraic Problem

We now take stock and summarize the conditions on the Mellin amplitude that follow from our discussion in the previous sections:

1. *Crossing symmetry:* As the external operators are identical bosonic operators, the Mellin amplitude $\mathcal{M}_k$ satisfies the crossing relations

$$
\begin{aligned}
\sigma^k \mathcal{M}_k(u,t;1/\sigma,\tau/\sigma) = \mathcal{M}_k(s,t,;\sigma,\tau) \\
\tau^k \mathcal{M}_k(t,s;\sigma/\tau,1/\tau) = \mathcal{M}_k(s,t;\sigma,\tau) .
\end{aligned}
\tag{4.82}
$$

2. *Analytic properties:* $\mathcal{M}_k$ has only simple poles in correspondence with the exchanged single-trace operators. Denoting the position of the simple poles in the s-, t- and u-channel as s_0, t_0, u_0, they are:

$$
\begin{aligned}
s_0 &= 4, 6, \ldots, 4k-2 , \\
t_0 &= 4, 6, \ldots, 4k-2 , \\
u_0 &= 4, 6, \ldots, 4k-2 .
\end{aligned}
\tag{4.83}
$$

   Moreover, the residue at any of the poles must be a polynomial in the other Mandelstam variable.

3. *Asymptotic behavior:* $\mathcal{M}_k$ should grow at most linearly in the asymptotic regime of large Mandelstam variables,

$$
\mathcal{M}_k(\beta s, \beta t, \beta u, \sigma, \tau) \sim O(\beta) , \qquad \beta \to \infty . \tag{4.84}
$$

4. *Superconformal symmetry:* $\mathcal{M}_k$ can be written in terms an auxiliary amplitude $\widetilde{\mathcal{M}}_k$ acted upon by the difference operator $\widehat{\Theta}$,

$$\mathcal{M}_k = \widehat{\Theta} \circ \widetilde{\mathcal{M}}_k \ , \tag{4.85}$$

where the action of $\widehat{\Theta}$ has been defined in the previous subsection.

The name of the game is to find a function $\widetilde{\mathcal{M}}_k(s,t;\sigma,\tau)$ such that all the conditions are simultaneously satisfied. We leave a detailed analysis of this very constrainted "bootstrap" problem for the future. As in the four-dimensional case analyzed in Section 4.1 and Section 4.2, we find it very plausible that this problem has a unique solution (up to overall rescaling).

4.4 Partial Solution for $AdS_7 \times S^4$

In this subsection we give solutions to the algebraic problem defined above for $k = 2, 3$. These solutions are obtained from the position space results, but look much simpler in Mellin space when the prescription (4.85) is implemented.

$k = 2$

We start from the simplest example of $k = 2$. In this case, the homogenous part $\mathcal{H}_2$ is a degree-0 polynomial of σ and τ. Therefore there is only one R-symmetry structure in the auxiliary amplitude $\widetilde{\mathcal{M}}_2$. The answer is given by

$$\widetilde{\mathcal{M}}_2(s,t;\sigma,\tau) = \frac{32}{n^3(s-6)(s-4)(t-6)(t-4)(\tilde{u}-6)(\tilde{u}-4)} \tag{4.86}$$

which is manifestly symmetric under the permutation of s, t and $\tilde{u}$.

$k = 3$

Moving on to the next simplest case of $k = 3$, we know that $\mathcal{H}_3$ is a degree-one polynomial of σ and τ and therefore consists of three terms. The three R-symmetry monomials σ, τ, 1 are in the same orbit under the action of the crossing symmetry group. Hence, $\widetilde{\mathcal{M}}_{3,010}$ and $\widetilde{\mathcal{M}}_{3,001}$ are related to $\widetilde{\mathcal{M}}_{3,100}$ via (4.73)

$$\widetilde{\mathcal{M}}_{3,010}(s,t) = \widetilde{\mathcal{M}}_{3,100}(\tilde{u},t), \qquad \widetilde{\mathcal{M}}_{3,001}(s,t) = \widetilde{\mathcal{M}}_{3,100}(t,s). \tag{4.87}$$

Finally, $\widetilde{\mathcal{M}}_{3,100}$ is given by

$$\widetilde{\mathcal{M}}_{3,100}(s,t) = \frac{8(s-7)}{3n^3(s-8)(s-6)(s-4)(t-10)(t-8)(\tilde{u}-10)(\tilde{u}-8)} \,, \tag{4.88}$$

and the full auxiliary Mellin amplitude is

$$a\widetilde{\mathcal{M}}_3(s,t;\sigma,\tau) = a\widetilde{\mathcal{M}}_{3,100}(s,t) + b\widetilde{\mathcal{M}}_{3,010}(s,t) + c\widetilde{\mathcal{M}}_{3,001}(s,t) \,. \tag{4.89}$$

Chapter 5

Superconformal Ward Identities for Mellin Amplitudes

In the previous chapter we showed how to translate the task of computing four-point functions into solving an algebraic bootstrap problem. A key step of this approach is to rewrite the position space *solution* to the superconformal Ward identity in Mellin space. However, the form of the position space solution is highly sensitive to the number of spacetime dimensions: in $d = 4$, it is simply given by (4.1); in $d = 6$, though (4.56) is schematically the same as (4.1), a very complicated differential operator Υ (4.57) replaced the role of the simple factor R (4.2). The situation becomes even worse in odd spacetime dimensions where non-local differential operators appear in the solution [37]. These non-local differential operators makes the meaning of the solution extremely obscure in Mellin space. On the other hand, the superconformal Ward identity (2.23)

$$(\chi\partial_\chi - \epsilon\alpha\partial_\alpha)\mathcal{G}(\chi,\chi';\alpha,\alpha')\big|_{\alpha=1/\chi} = 0 \; . \tag{5.1}$$

takes a universal form for d where the dependence on the spacetime dimension only enters in the factor $\epsilon = \frac{d}{2} - 1$ in the identity. This motivates us to translate only the superconformal Ward *identity* itself into Mellin space. The upshot of this rewriting is that we obtain a set of difference identities of the Mellin amplitudes which should be viewed as the Mellin amplitude superconformal Ward identities. We present this d-independent treatment for imposing superconformal constraints on Mellin amplitudes in Section 5.1. Then in Section 5.1.1 we discuss how to use this technique to bootstrap holographic Mellin amplitudes. We apply this method to a number of examples in Section 5.2, among which the stress tensor four-point function in $AdS_4 \times S^7$ is of special interest. The dual theory is a canonical example of 3d SCFT, namely, ABJM theory at $k = 1$. This correlator was not accessible using any prior method, and was computed in the first time in 20 years [6] using the method of this chapter.

5.1 Translating the Position Space Superconformal Ward Identity into Mellin Space

From the Mellin space point of view the superconformal Ward identity (2.23) seems rather unappealing at first sight. Our ideal scenario is to have factors in the form of $U^m V^n$ multiplying an inverse Mellin transformation. We can absorb such factors by shifting the s, t variables and trade them for difference operators that act on the integrand. However, the variables χ and χ' appear asymmetrically on the left side of the identity (2.23). If one naively solved χ and χ in terms of U, V, one would encounter square roots in these variables, making how to proceed unclear.

We now offer in this section a simple observation. This observation allows us to obtain relations in the Mellin amplitude from the position space identity (2.23), and these relations constitute the superconformal Ward identities in Mellin space. For starters, let us write the differential operator $\chi\partial_\chi$ as

$$\chi\frac{\partial}{\partial\chi} = U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V} - \frac{1}{1-\chi}V\frac{\partial}{\partial V} \ . \tag{5.2}$$

We act this operator on $\mathcal{G}_{\rm conn}(U,V;\sigma,\tau) = \sum_{L+M+N=\mathcal{L}} \sigma^M \tau^N \mathcal{G}_{{\rm conn},LMN}(U,V)$ but do not evaluate the action of $U\partial_U$ and $V\partial_V$ on $\mathcal{G}_{{\rm conn},LMN}(U,V)$ at this stage. For the R-symmetry part, acting with $\alpha\partial_\alpha$ and then setting $\alpha = 1/\chi$

turn the monomials $\sigma^i\tau^j$ into some simple rational functions

$$
\begin{aligned}
1 &\quad\rightarrow\quad 0\,, \\
\sigma &\quad\rightarrow\quad \left(\frac{1}{\chi}\right)\alpha\,, \\
\tau &\quad\rightarrow\quad \left(\frac{1}{\chi}\right)\alpha' - \frac{1}{\chi}\,, \\
\sigma^2 &\quad\rightarrow\quad \left(\frac{2}{\chi^2}\right)\alpha'^2\,, \\
\sigma\tau &\quad\rightarrow\quad \left(\frac{2-\chi}{\chi^2}\right)\alpha'^2 - \left(\frac{2-\chi}{\chi^2}\right)\alpha'\,, \\
\tau^2 &\quad\rightarrow\quad \left(\frac{2(1-\chi)}{\chi^2}\right)\alpha'^2 + \left(\frac{4(1-\chi)}{\chi^2}\right)\alpha' + \left(\frac{2(1-\chi)}{\chi^2}\right)\,, \\
&\cdots\cdots
\end{aligned}
\tag{5.3}
$$

Performing the twist $\alpha = 1/\chi$ alone on $\sigma^i\tau^j$ also produces similar rational functions. We notice that the highest power of α in $\mathcal{G}$ is $\mathcal{L}$, as it follows from the fact that $\mathcal{G}$ is a degree-$\mathcal{L}$ polynomial of σ and τ. It is easy to see that the action of these operations does not change this degree. This instructs us to take out a factor $(1-\chi)^{-1}\chi^{-\mathcal{L}}$ from (2.23), so that the left side becomes a degree-$(\mathcal{L}+1)$ polynomial of χ. Schematically, we can write the new identity as

$$f_0 + \chi f_1 + \chi^2 f_2 + \ldots + \chi^{\mathcal{L}+1} f_{\mathcal{L}+1} = 0 \tag{5.4}$$

where $f_i = f_i(U, V; \alpha')$ are functions of the conformal cross ratios U, V and the untwisted R-symmetry variable α'. Note that an ambiguity exists in the change of variables (2.22), namely, under the exchange of $\chi \leftrightarrow \chi'$ the variables U and V remain the same. Hence by exchanging χ with χ' we get from (5.4) another copy of the identity for free

$$f_0 + \chi' f_1 + \chi'^2 f_2 + \ldots + \chi'^{\mathcal{L}+1} f_{\mathcal{L}+1} = 0\,. \tag{5.5}$$

Taking the sum of these two identities, we arrive at the following equation

$$2f_0 + (\chi + \chi')f_1 + (\chi^2 + \chi'^2)f_2 + \ldots + (\chi^{\mathcal{L}+1} + \chi'^{\mathcal{L}+1})f_{\mathcal{L}+1} = 0\,. \tag{5.6}$$

Crucially, the appearance of χ and χ' is now symmetrized and each $\chi^n + \chi'^n$ can be rewritten as a finite linear combination of $U^m V^n$[1]. After making this

[1]This is easy to see by induction:

$$\chi^n + \chi'^n = (\chi^{n-1} + \chi'^{n-1})(\chi + \chi') - \chi\chi'(\chi^{n-2} + \chi'^{n-2})\,. \tag{5.7}$$

replacement, we can now exploit the superconformal Ward identity in Mellin space using elementary manipulations of the inverse Mellin transformation.

Substituting $\mathcal{G}_{\text{conn},LMN}(U,V)$ with its inverse Mellin representation,

$$\begin{aligned}\mathcal{G}_{\text{conn},LMN}(U,V) = \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i}\frac{dt}{4\pi i}\, U^{\frac{s}{2}-\frac{\epsilon(k_3+k_4)}{2}+\epsilon\mathcal{L}}\, V^{\frac{t}{2}-\frac{\epsilon\min\{k_1+k_4,k_2+k_3\}}{2}} \\ \times \mathcal{M}_{LMN}(s,t)\Gamma_{k_1k_2k_3k_4}\ ,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\Gamma_{k_1k_2k_3k_4} \equiv {} & \Gamma[-\frac{s}{2}+\frac{\epsilon(k_1+k_2)}{2}]\Gamma[-\frac{s}{2}+\frac{\epsilon(k_3+k_4)}{2}]\Gamma[-\frac{t}{2}+\frac{\epsilon(k_2+k_3)}{2}] \\ & \times\Gamma[-\frac{t}{2}+\frac{\epsilon(k_1+k_4)}{2}]\Gamma[-\frac{u}{2}+\frac{\epsilon(k_1+k_3)}{2}]\Gamma[-\frac{u}{2}+\frac{\epsilon(k_2+k_4)}{2}]\ ,\end{aligned} \tag{5.9}$$

the following dictionary then becomes clear

$$U\frac{\partial}{\partial U} \quad \Rightarrow \quad \left[\frac{s}{2}-\frac{\epsilon(k_3+k_4)}{2}+\epsilon\mathcal{L}\right]\times\ , \tag{5.10}$$

$$V\frac{\partial}{\partial V} \quad \Rightarrow \quad \left[\frac{t}{2}-\frac{\epsilon\min\{k_1+k_4,k_2+k_3\}}{2}\right]\times\ , \tag{5.11}$$

$$U^mV^n \quad \Rightarrow \quad \text{shift } s \text{ by } -2m \text{ and } t \text{ by } -2n. \tag{5.12}$$

Note the shifts in the third line act on the *reduced* Mellin amplitude, *i.e.*, both the Mellin amplitude and the Gamma functions. It becomes more convenient if we preserve in each integrand a common factor of Gamma functions $\Gamma_{k_1k_2k_3k_4}$ as defined in (5.9) when we add up the inverse Mellin transformations. The monomial U^mV^n then becomes an operator $\underline{\widehat{U^mV^n}}$[2] that only acts on the *Mellin amplitude* in the following way

$$U^mV^n \Rightarrow \underline{\widehat{U^mV^n}} \tag{5.13}$$

$$\begin{aligned}\underline{\widehat{U^mV^n}}\circ\mathcal{M}(s,t) = {} & \mathcal{M}(s-2m,t-2n)\left(\frac{\epsilon(k_1+k_2)-s}{2}\right)_m\left(\frac{\epsilon(k_3+k_4)-s}{2}\right)_m \\ & \times\left(\frac{\epsilon(k_1+k_4)-t}{2}\right)_n\left(\frac{\epsilon(k_2+k_3)-t}{2}\right)_n\left(\frac{\epsilon(k_1+k_3)-u}{2}\right)_{-m-n} \\ & \times\left(\frac{\epsilon(k_2+k_4)-u}{2}\right)_{-m-n}\ .\end{aligned} \tag{5.14}$$

Note $\chi\chi' = U$ and for $n=1$, $\chi+\chi' = U-V+1$.

[2] This operator should not be confused with the operator $\widehat{U^mV^n}$ used in Chapter 4. We put an underline to distinguish it.

Here $(a)_n$ is the Pochhammer symbol. Since (5.6) is a degree-$\mathcal{L}$ polynomial of α', we get in total $\mathcal{L}+1$ identities for the Mellin amplitude $\mathcal{M}(s,t;\sigma,\tau)$. These are the Mellin space superconformal Ward identities.

To close this subsection, let us summarize the procedure for implementing the superconformal Ward identity in Mellin space:

1. We start with $(\chi\partial_\chi - \epsilon\alpha\partial_\alpha)\mathcal{G}(\chi,\chi';\alpha,\alpha')$. $\mathcal{G}(\chi,\chi';\alpha,\alpha')$ is decomposed into R-symmetry monomials $\sum_{L+M+N=\mathcal{L}}\sigma^M\tau^N\mathcal{G}_{\mathrm{conn},LMN}(U,V)$ and σ and τ are related to α, α' via (2.22). We write the action of $\chi\partial_\chi$ as (5.2) and perform the action of $\alpha\partial_\alpha$. The action of $U\partial_U$ and $V\partial_V$ on $\mathcal{G}_{\mathrm{conn},LMN}(U,V)$ are not evaluated at this step.

2. We perform the twist $\alpha=1/\chi$ and multiply the expression with $(1-\chi)\chi^{\mathcal{L}}$ to make it a polynomial of χ.

3. We replace all χ with χ' and add up the two expressions. All the χ and χ' are then rewritten as polynomials of U and V.

4. We use the inverse Mellin representations of the correlator. This amounts to replacing each $G_{\mathrm{conn},LMN}$ with $\mathcal{M}_{LMN}$. There are additional factors and derivatives of U and V. For $U\partial_U$ and $V\partial_V$, we replace them with the factors (5.10) and (5.11) that multiply the Mellin amplitude. For monomials U^mV^n, we replace them with the operator $\underline{\widehat{U^mV^n}}$ whose action on the Mellin amplitude is given by (5.14).

5. We organize the expression by powers of α'. All the polynomial coefficients of α' are linear functions of $\mathcal{M}_{LMN}$ with shifted arguments, and they are required to be zero. These equations are the Mellin space superconformal Ward identities.

5.1.1 Bootstrapping Holographic Mellin Amplitudes

Now we are equipped with the Mellin space superconformal Ward identities, we can formulate another bootstrap-inspired approach which computes holographic correlators entirely within the Mellin space. The idea is straightforward: we formulate an ansatz for the Mellin amplitude and then solve the ansatz using superconformal symmetry. As was reviewed in Section 3.1, the structure of supergravity Mellin amplitudes is very simple. As a function of the Mandelstam variables s and t, the Mellin amplitude splits into a singular part and a regular part. The singular part has only simple poles in s,

t and u, whose locations are determined by the spectrum of the theory and the cubic coupling selection rules. At each simple pole, the residue has to be a polynomials of the other independent Mandelstam variable – it is the linear superposition of Mack polynomials. Because we have in the spectrum particles with maximal spin two, the degree of such a residue polynomial is bounded to be two. The regular part is even simpler. It is a linear polynomial of the Mandelstam variables as is required by the consistency with the flat space limit. Of course each term also depends polynomially on the R-symmetry variables σ and τ, with a degree $\mathcal{L}$ determined by the weights k_i of the external operators. Computing the Mellin amplitudes amounts to fixing the coefficients in the residues and in the regular part, and the Mellin space superconformal Ward identities in Section 5.1 help us achieve precisely that.

Let us now divide the further description of the ansatz into two scenarios, depending on the number of simple poles in the singular part of the Mellin amplitude is finite or infinite. The former situation occurs in $AdS_5 \times S^5$ and $AdS_7 \times S^4$. In this case we can use a totally general ansatz: we do not make any specification of the coefficients in the residue polynomials and the regular part, and they are left as unknowns to be solved. When the four external operators are identical, the Mellin amplitude ansatz is further required to have crossing symmetry. This is the most general ansatz one can write down that is compatible with the qualitative information of the bulk supergravity. Applying and solving the Mellin space superconformal Ward identities are completely straightforward as it is a finite problem. For the latter scenario, such as in $AdS_4 \times S^7$, proceeding with such a generic ansatz appears to be technically involved because we need infinitely many coefficients to parameterize the ansatz. We simplify the problem by using an ansatz parallel to the ones used in the position space method. To be precise, we will use the explicit Mellin amplitudes of the exchange Witten diagrams (but only the singular part) and write the singular part of the ansatz as a linear combination of such exchange Mellin amplitudes. These amplitudes are not hard to obtain because it is known that the Mellin amplitude of a conformal block with the same quantum numbers of the exchanged single-trace operator has the same pole and same residues. The Mellin amplitude of conformal blocks can be found in, *e.g.*, [40, 45]. For the regular part, we will use the same general parameterization as in the former case. Such an ansatz is general enough to encompass the correct answer, but does not have as much power to exclude other regions in the space of ansatz as it did for

$AdS_5 \times S^5$ and $AdS_7 \times S^4$.

5.2 Applications

5.2.1 Stress Tensor Four-Point Functions for $AdS_5 \times S^5$

As a simplest application, we reproduce in this subsection the four-point function $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$ for IIB supergravity on $AdS_5 \times S^5$. From the selection rules we know that the only contributing fields in the exchange Witten diagrams are: the scalar field itself, a vector field with $\Delta = 3$ and a graviton field with $\Delta = 4$. All these fields have coinciding conformal twist $\tau = 2$. The simple poles in their Mellin amplitudes should truncate to just a single one at 2. Hence the singular part of the ansatz should consist of the following terms

$$\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u \tag{5.15}$$

where

$$\mathcal{M}_s = \sum_{\substack{0 \le i,j \le 2, \\ 0 \le i+j \le 2}} \sum_{0 \le a \le 2} \frac{\lambda^{(s)}_{ij;a}\, \sigma^i \tau^j t^a}{s-2} , \tag{5.16}$$

$$\mathcal{M}_t = \sum_{\substack{0 \le i,j \le 2, \\ 0 \le i+j \le 2}} \sum_{0 \le a \le 2} \frac{\lambda^{(t)}_{ij;a}\, \sigma^i \tau^j u^a}{t-2} , \tag{5.17}$$

$$\mathcal{M}_u = \sum_{\substack{0 \le i,j \le 2, \\ 0 \le i+j \le 2}} \sum_{0 \le a \le 2} \frac{\lambda^{(u)}_{ij;a}\, \sigma^i \tau^j s^a}{u-2} , \tag{5.18}$$

and $s + t + u = 8$. We should also add to ansatz the following polynomial term that represents the contact interactions

$$\mathcal{M}_c = \sum_{\substack{0 \le i,j \le 2, \\ 0 \le i+j \le 2}} \sum_{\substack{0 \le a,b \le 1, \\ 0 \le a+b \le 1}} \mu_{ij;ab}\, \sigma^i \tau^j s^a t^b . \tag{5.19}$$

The most general ansatz in the supergravity result is therefore

$$\mathcal{M}_{\text{ansatz}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c , \tag{5.20}$$

and it should satisfy in addition the following crossing equations

$$\mathcal{M}_{\text{ansatz}}(s,t;\sigma,\tau)=\tau^2\mathcal{M}_{\text{ansatz}}(t,s;\sigma/\tau,1/\tau)=\sigma^2\mathcal{M}_{\text{ansatz}}(u,t;1/\sigma,\tau/\sigma)\ . \tag{5.21}$$

We now impose the Mellin space superconformal Ward identities following the procedure given in Section 5.1. We obtain three sets of equations of $\mathcal{M}_{ansatz}$ with shifted arguments, as (5.6) is degree-two in α' and the coefficients of α'^n must separately vanish. Bringing all terms in each equation to the minimal common denominator, the numerator is a polynomial in s and t with coefficients linearly depending on the unfixed parameters. Requiring these coefficients to vanish gives us a set of linear equations.

We solve these equations together with the crossing equations and we arrive at the following solution

$$\begin{aligned}\mathcal{M}=&C\frac{\sigma(4u-4t+8)+\tau(4t-4u+8)+(u^2+t^2-6u-6t+16)}{s-2}\\&+C\frac{\tau(\sigma(4u-4s+8)+\tau(s^2+u^2-6s-6u+16)+(4s-4u+8))}{t-2}\\&+C\frac{\sigma(\sigma(s^2+t^2-6s-6t+16)+\tau(4t-4s+8)+(4s-4t+8))}{u-2}\\&+C(-s-u\sigma^2-t\tau^2+4(t+u-2)\sigma\tau+4(s+u-2)\sigma+4(s+t-2)\tau)\end{aligned} \tag{5.22}$$

where C is an unfixed overall coefficient. This answer agrees with the original supergravity result [17].

5.2.2 Next-Next-to-Extremal Four-Point Functions for $AdS_7\times S^4$

In this section we apply the Mellin space method to compute next-next-to-extremal correlators for the eleven dimensional supergravity compactified on $AdS_7\times S^4$. The *extremality* of a four-point function is defined by

$$E=k_1-k_2-k_3-k_4 \tag{5.23}$$

where the ordering of the weights $k_1\geq k_2\geq k_3\geq k_4$ is assumed. R-symmetry selection rules determine that E is an even integer. When $E=0,2$, the four-point functions are respectively said to be *extremal* and *next-to-extremal*. For

$\mathcal{N} = 4$ SYM in 4D, such correlators are protected by non-renormalization theorems [33, 85, 86, 87, 88] and they have the same value as in the free limit of the theory. For 6D (2,0) theories, the notion of "non-renormalization" is moot due to the absence of an exactly marginal coupling. However, one can still argue from the finiteness of the boundary correlator that the supergravity couplings should vanish as the relevant Witten diagrams are divergent [89]. A regularization procedure needs to be taken, and the end result is that the correlators computed from supergravity are rational functions of the cross ratios – the dependence on the cross ratios is similar to that in a generalized free field theory. The $E = 4$ case is the first case free of such subtleties and the four-point function starts to depend on the cross ratios in a more non-trivial manner. Such *next-next-to-extremal* correlators will be the focus of this section. We will take two operators with $k_3 = k_4 = k + 2$ and the other two operators with $k_1 = n+k$, $k_2 = n-k$. The same class of four-point functions have been studied for IIB $AdS_5 \times S^5$ supergravity in [21] and we will find that the solution for $AdS_7 \times S^4$ takes a very similar form.

Let us start by writing down an ansatz for next-next-to-extremal correlators for $AdS_7 \times S^4$. From the selection rules of the cubic couplings, we find that only the following fields in Table 2.2 can be exchanged in the s-channel: the scalar field s_{2k+2}, the vector field $A_{\mu,2k+2}$ and the massive symmetric tensor field $\varphi_{\mu\nu,2k+2}$. These three fields all have the same conformal twist $\tau = 4k + 4$. The Mellin amplitude should therefore have a leading simple pole at $s = 4k + 4$ and a satellite pole at $s = 4k + 6$. Moreover, since the maximal spin of the exchanged fields is two, we anticipate that the residues at these simple poles are degree-two polynomials in the other Mandelstam variable t. The following terms therefore should be part of the ansatz for the total Mellin amplitude

$$\mathcal{M}_s(s,t;\sigma,\tau) = \sum_{\substack{0\leq i,j\leq 2,\\ 0\leq i+j\leq 2}} \sum_{0\leq a\leq 2} \frac{\lambda^{(s,1)}_{ij;a}\, \sigma^i\tau^j t^a}{s-(4k+4)} + \sum_{\substack{0\leq i,j\leq 2,\\ 0\leq i+j\leq 2}} \sum_{0\leq a\leq 2} \frac{\lambda^{(s,2)}_{ij;a}\, \sigma^i\tau^j t^a}{s-(4k+6)}\,. \tag{5.24}$$

Notice that in the residues in the above ansatz we have left the dependence on the R-symmetry variables and t completely arbitrary, apart from the bounded degrees.

Similarly, for t and u-channel, we also have three types of field exchanges: s_{2n}, $A_{\mu,2n}$ and $\varphi_{\mu\nu,2n}$. The three fields have the same conformal twist $\tau = 2n$ and maximal spin two, and therefore lead to simple poles at $t = 2n$, $t = 2n+2$,

$u = 2n$ and $u = 2n+2$. The residues at these simple poles are also degree-two polynomials. We therefore further include the following $\mathcal{M}_t$ and $\mathcal{M}_u$

$$\mathcal{M}_t(s,t;\sigma,\tau) = \sum_{\substack{0\leq i,j\leq 2,\\ 0\leq i+j\leq 2}} \sum_{0\leq a\leq 2} \frac{\lambda^{(t,1)}_{ij;a}\ \sigma^i\tau^j u^a}{t-2n} + \sum_{\substack{0\leq i,j\leq 2,\\ 0\leq i+j\leq 2}} \sum_{0\leq a\leq 2} \frac{\lambda^{(t,2)}_{ij;a}\ \sigma^i\tau^j u^a}{t-(2n+2)}\ , \tag{5.25}$$

$$\mathcal{M}_u(s,t;\sigma,\tau) = \sum_{\substack{0\leq i,j\leq 2,\\ 0\leq i+j\leq 2}} \sum_{0\leq a\leq 2} \frac{\lambda^{(u,1)}_{ij;a}\ \sigma^i\tau^j s^a}{u-2n} + \sum_{\substack{0\leq i,j\leq 2,\\ 0\leq i+j\leq 2}} \sum_{0\leq a\leq 2} \frac{\lambda^{(u,2)}_{ij;a}\ \sigma^i\tau^j s^a}{u-(2n+2)}\ , \tag{5.26}$$

with $u = 4(n+k+2) - s - t$.

This has exhausted all the poles in the Mellin amplitude. Additionally we should also have the following polynomial piece to account for the contact interactions

$$\mathcal{M}_c(s,t;\sigma,\tau) = \sum_{\substack{0\leq i,j\leq 2,\\ 0\leq i+j\leq 2}} \sum_{\substack{0\leq a,b\leq 1,\\ 0\leq a+b\leq 1}} \mu_{ij;ab}\ \sigma^i\tau^j s^a t^b\ . \tag{5.27}$$

The full general ansatz is then the sum of the four parts

$$\mathcal{M}_{\text{ansatz}} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_c\ . \tag{5.28}$$

Now we impose the Mellin space superconformal Ward identity. Note $\mathcal{M}_{ansatz}$ has finitely many terms and contains only finitely many unknown parameters. The problem is therefore completely finite and elementary. Solving these constraints is very straightforward and we find the answer is unique up to an overall constant. For arbitrary n and k, we find the following solution

$$\begin{aligned}\mathcal{M}(s,t;\sigma,\tau) = {}& \mathcal{M}_1(s,t) + \sigma^2\mathcal{M}_{\sigma^2}(s,t) + \tau^2\mathcal{M}_{\tau^2}(s,t) + \sigma\mathcal{M}_\sigma(s,t) \\ & + \tau\mathcal{M}_\tau(s,t) + \sigma\tau\mathcal{M}_{\sigma\tau}(s,t)\ ,\end{aligned} \tag{5.29}$$

with the R-symmetry partial amplitudes given by

$$
\begin{aligned}
\mathcal{M}_1(s,t) &= -C_{n,k}\frac{(k+1)(4k+2n-t+4)(4k-2n+t)}{8(s-4k-4)} \\
&\quad - \frac{C_{n,k}}{32}(2k+2n+1)(4k+2n-t+4) \\
&\quad + C_{n,k}\frac{(2k-2n+3)(4k+2n-t+4)(4k-2n+t+2)}{32(s-4k-6)}\ , \\
\mathcal{M}_{\sigma^2}(s,t) &= C_{n,k}\frac{n(4k+2n-t+4)(-4k+2n+t-8)}{16(s+t-4k-2n-8)} \\
&\quad - \frac{C_{n,k}}{32}(2k+2n+1)(4k+2n-t+4) \\
&\quad - C_{n,k}\frac{(2k+1)(4k-2n-t+6)(4k+2n-t+4)}{32(s+t-4k-2n-6)}\ , \\
\mathcal{M}_{\tau^2}(s,t) &= -C_{n,k}\frac{n(s-4)(4n-s)}{16(t-2n)} - C_{n,k}\frac{(2k+1)(s-2)(4n-s)}{32(t-2n-2)} \\
&\quad - \frac{C_{n,k}}{32}(2k+2n+1)(4n-s)\ ,
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
\mathcal{M}_{\sigma}(s,t) =& C_{n,k}\frac{(k+1)(4k+3)(4k+2n-t+4)}{4(s-4k-4)} - C_{n,k}\frac{(2k+1)n(4k+2n-t+4)}{8(s+t-4k-2n-6)} \\
& - C_{n,k}\frac{(k+1)(2k-2n+3)(4k+2n-t+4)}{4(s-4k-6)} \\
& - C_{n,k}\frac{n(2n-1)(4k+2n-t+4)}{8(s+t-4k-2n-8)} + \frac{C_{n,k}}{16}(2k+2n+1)(4k+2n-t+4)\ , \\
\mathcal{M}_{\tau}(s,t) =& C_{n,k}\frac{(k+1)(4k+3)(4k-2n+t)}{4(s-4k-4)} + C_{n,k}\frac{(2k+1)n(s-2)}{8(t-2n-2)} \\
& + C_{n,k}\frac{n(2n-1)(s-4)}{8(t-2n)} - C_{n,k}\frac{(k+1)(2k-2n+3)(4k-2n+t+2)}{4(s-4k-6)} \\
& + \frac{C_{n,k}}{16}(2(k+n)(4k+s+t)-2n+s+t-4) \\
\mathcal{M}_{\sigma\tau}(s,t) =& C_{n,k}\frac{n(2n-1)(4n-s)}{8(t-2n)} + C_{n,k}\frac{(2k+1)n(4n-s)}{8(t-2n-2)} \\
& - C_{n,k}\frac{(2k+1)n(-4k+2n+t-6)}{8(s+t-4k-2n-6)} - C_{n,k}\frac{n(2n-1)(-4k+2n+t-8)}{8(s+t-4k-2n-8)} \\
& + \frac{C_{n,k}}{16}(-2s(k+n)+4n(3k+3n+1)-s)\ .
\end{aligned}
\tag{5.31}
$$

The number $C_{n,k}$ above is an overall normalization that depends on the value of n, k and cannot be fixed by the symmetry considerations alone.

This remaining parameter can be however determined by using, *e.g.*, the three-point function of the three scalar fields.

The next-next-to-extremal four-point Mellin amplitude has a hidden simplicity. Using the prescription from Section 4.3.1, we find that the next-next-to-extremal Mellin amplitude can be written into the following remarkably simple one-term expression in terms of the auxiliary Mellin amplitude

$$\widetilde{\mathcal{M}}(s,t)=\frac{C_{n,k}}{(s-4k-6)(s-4k-4)(t-2n-2)(t-2n)(\tilde{u}-2n-2)(\tilde{u}-2n)}\,. \tag{5.32}$$

Translating the above result into position space, we find $\mathcal{H}$ consists of only one single $\bar{D}$-function

$$\mathcal{H}=64\;C_{n,k}U^{2n-2k+1}V^{-1}\bar{D}_{2n+2k+3,2n-2k-1,2k+3,2k+3}\,. \tag{5.33}$$

This is very similar to the $AdS_5\times S^5$ case [22], for which the next-next-to-extremal four-point functions have $\mathcal{H}=\tilde{C}_{n,k}U^{n-k}V^{-1}\bar{D}_{n+k+2,n-k,k+2,k+2}$ and $\tilde{C}_{n,k}$ is some constant.

When we take the special value of $n=2$, $k=0$ in the above results, the four-point function becomes the equal-weight stress-tensor multiplet correlator $\langle\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$. This special case matches precisely our earlier result (4.86).

5.2.3 Stress Tensor Four-Point Functions for $AdS_4\times S^7$

In this section, we compute the holographic one-half BPS four-point function of operators with $k_i=2$ from eleven dimensional supergravity compactified on $AdS_4\times S^7$ at tree-level. The supergravity theory is conjectured to be dual to an $\mathcal{N}=8$ SCFT in three dimensions which describes the infrared limit of the effective theory on a large number N of coincident $M2$-branes in flat space. An explicit realization of this effective theory is the ABJM theory [26] with Chern-Simons level $k=1$ and large N. The theory has $OSp(4|8)$ superconformal symmetry[3] which includes the conformal symmetry group $SO(3,2)$ and R-symmetry group $SO(8)$ as bosonic subgroups.

The one-half BPS operator $\mathcal{O}^{IJ}_{k=2}$ is the superconformal primary of the $\mathcal{N}=8$ stress-tensor multiplet. It has conformal dimension $\Delta=1$ and transforms as the symmetric-traceless representation $\mathbf{35}_c$ (the rank-two symmetric

[3]The full $\mathcal{N}=8$ superconformal symmetry of $k=1$ ABJM theory is not manifest at the classical level, but an enhancement from $\mathcal{N}=6$ to $\mathcal{N}=8$ is anticipated at the quantum level from string theory arguments.

traceless product of the $\mathbf{8}_c$) under the $SO(8)$ R-symmetry group. In the bulk supergravity dual, it corresponds to a scalar field in AdS_4 with squared mass $m^2 = -2$. From the supergravity selection rules, we find that only three fields can be exchanged in each channel, namely, the same scalar field itself, a vector field with dimension $\Delta = 2$ in the representation $\mathbf{28}$ and a graviton field with dimension $\Delta = 3$ in the singlet representation. However all the three fields have conformal twist $\tau = 1$, the truncation condition of exchange Witten diagrams therefore cannot be met and the Mellin amplitude will have an infinite series of poles. In the position space language, this means that the "without really trying" method of [36] is no longer effective and the exchange diagrams can only be expressed as a infinite sum of D-functions.

This property of the exchange diagrams presents some technical challenge even when we are working directly in Mellin space. Since the series of simple poles in the three channels do not truncate, we would need infinitely many parameters to parameterize the residues if we were to work with an ansatz such as used in the AdS_5 and AdS_7 cases. We postpone the analysis with such a general ansatz and take a more restrictive ansatz where the Mellin amplitudes of the exchange Witten diagrams are used. In total, the ansatz is a crossing-symmetric sum of the exchange Mellin amplitudes plus contact Mellin amplitudes. This reduces the variable coefficients to a finite set, with three of them tracking the contribution of the scalar, vector and graviton exchanges, and a few more for the contact diagrams. Such an ansatz is precisely what was used in the AdS_5 and AdS_7 position space method in Section 2.3. Note that the same ansatz for AdS_4 does not give us much mileage in position space because of the difficulty to handle an infinite sum of D-functions. However, taking advantage of the simple structure of the Mellin amplitudes, it is possible to obtain a closed form answer by solving the Mellin space version of the superconformal Ward identities. Let us now spell out the details.

As mentioned above, our ansatz for the full Mellin amplitude is a sum of the exchange diagram amplitudes and contact diagram amplitudes

$$\mathcal{M}(s,t;\sigma,\tau) = \mathcal{M}_{\text{s-exchange}} + \mathcal{M}_{\text{t-exchange}} + \mathcal{M}_{\text{u-exchange}} + \mathcal{M}_{\text{contact}} \,. \tag{5.34}$$

The s-channel exchange amplitude $\mathcal{M}_{\text{s-exchange}}$ is comprised of the amplitudes from exchanging three fields

$$\mathcal{M}_{\text{s-exchange}} = \lambda_g \mathcal{M}_{\text{graviton}}(s,t) + \lambda_v(\sigma-\tau)\mathcal{M}_{\text{vector}}(s,t) + \lambda_s(4\sigma+4\tau-1)\mathcal{M}_{\text{scalar}}(s,t) \tag{5.35}$$

where the factors 1, $(\sigma-\tau)$, $(4\sigma+4\tau-1)$ are the R-symmetry polynomials associated with the irreducible representations of the exchanged fields. The numbers λ_g, λ_v and λ_s are the unknown coefficients that need to be fixed. Generally, the Mellin amplitude of an exchange Witten diagram contains a singular part which is a sum of simple poles and a regular part which is a polynomial. As was alluded to before, the singular part of the exchange Mellin amplitude has the same poles and residues as the Mellin space expression of an conformal block whose conformal dimension and spin are identical to those of the exchanged single-trace operator. On the other hand, since we have a contact part $\mathcal{M}_{\text{contact}}$ in the ansatz, we do not need to keep track of the polynomial piece in the exchange amplitudes. Such polynomial terms can just be swept into $\mathcal{M}_{\text{contact}}$ with a redefinition of the parameters. Therefore we only write down the singular pieces in $\mathcal{M}_{\text{graviton}}$, $\mathcal{M}_{\text{vector}}$, $\mathcal{M}_{\text{scalar}}$, and using the expressions in [45] we have

$$\begin{aligned}
\mathcal{M}_{\text{graviton}} &= \sum_{n=0}^{\infty} \frac{3\sqrt{\pi}\cos[n\pi]\Gamma[-\frac{3}{2}-n]}{4n!\Gamma[\frac{1}{2}-n]^2} \\
&\quad\times \frac{4n^2-8ns+8n+4s^2+8st-20s+8t^2-32t+35}{s-(2n+1)} \,, \\
\mathcal{M}_{\text{vector}} &= \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\cos[n\pi]}{(1+2n)\Gamma[\frac{1}{2}-n]\Gamma[1+n]} \frac{2t+s-4}{s-(2n+1)} \,, \\
\mathcal{M}_{\text{scalar}} &= \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\cos[n\pi]}{n!\Gamma[\frac{1}{2}-n]} \frac{1}{s-(2n+1)} \,.
\end{aligned}$$

In the above expressions we have appropriately symmetrized these amplitudes such that the residues in $\mathcal{M}_{\text{graviton}}$ and $\mathcal{M}_{\text{scalar}}$ are symmetric under exchanging $t \leftrightarrow u$ while the residues of $\mathcal{M}_{\text{vector}}$ are antisymmetric. Here, $u=4-s-t$. These symmetry properties follow from the usual four-point amplitude kinematics. The t-channel and u-channel exchange amplitudes are related to the s-channel amplitude by crossing symmetry

$$\begin{aligned}
\mathcal{M}_{\text{t-exchange}}(s,t;\sigma,\tau) &= \tau^2\mathcal{M}_{\text{s-exchange}}(t,s;\sigma/\tau,1/\tau) \,, && (5.36) \\
\mathcal{M}_{\text{u-exchange}}(s,t;\sigma,\tau) &= \sigma^2\mathcal{M}_{\text{s-exchange}}(u,t;1/\sigma,\tau/\sigma) \,. && (5.37)
\end{aligned}$$

For the contact contribution $\mathcal{M}_{\text{contact}}$, we have the ansatz

$$\mathcal{M}_{\text{contact}}(s,t;\sigma,\tau) = \sum_{\substack{0\le i,j\le 2,\\ 0\le i+j\le 2}} \sum_{\substack{0\le a,b\le 1,\\ 0\le a+b\le 1}} \mu_{ij;ab}\,\sigma^i\tau^j s^a t^b \,, \qquad (5.38)$$

together with the crossing symmetry condition that

$$\mathcal{M}_{\rm contact}(s,t;\sigma,\tau)=\tau^2\mathcal{M}_{\rm contact}(t,s;\sigma/\tau,1/\tau)=\sigma^2\mathcal{M}_{\rm contact}(u,t;1/\sigma,\tau/\sigma)\ . \tag{5.39}$$

The fact that $\mathcal{M}_{\rm contact}$ has degree two in σ and τ is in agreement with the fact that $\mathcal{L}=2$, and its linear dependence on s and t is required by the flat space limit.

We now plug this ansatz into the Mellin space Ward identities. Solving these identities now becomes a bit nontrivial compared to the previous cases because we have infinitely many poles. Some observation can be made which allows us to solve all the coefficients in two steps. First notice all the poles of s, t and u at odd integer positions can only come from the exchange part. This is because the shift operations always shift the arguments by an even integer amount and the parity of poles are preserved. Requiring those poles to vanish gives

$$\lambda_v=-4\lambda_s\ ,\qquad \lambda_g=\frac{\lambda_s}{3}\ . \tag{5.40}$$

These values are in agreement with the 3d $\mathcal{N}=8$ superconformal block [90][4]. To relate to the contact term parameters $\mu_{ij;ab}$, we need to look at the poles in the superconformal Ward identities[5] at $s+t=4,6,8$. Such poles come from the shifted Gamma functions in the reduced Mellin amplitudes. It is easiest if we first pick a specific n for $\mathcal{M}_{\rm graviton}$, $\mathcal{M}_{\rm vector}$, $\mathcal{M}_{\rm scalar}$ and obtain the residues. Then we resum in n to get an expression, which needs to be canceled with the term coming from the contact part $\mathcal{M}_{\rm contact}$. Requiring that and the crossing identity (5.39), we obtain enough equations to solve all the $\mu_{ij;ab}$ coefficients with respect to λ_s. Plugging this solution into $\mathcal{M}_{\rm contact}$,

---

[4]To see this, let us take the residue of the leading pole at $s=1$ and perform the t-integral. The scalar contributes to the four-point function by $-2\pi^3\lambda_s U^{\frac{1}{2}}g^{\rm coll}_{1,0}(V)$. Here $g^{\rm coll}_{\Delta,\ell}(V)$ is the collinear block

$$g^{\rm coll}_{\Delta,\ell}(V)\equiv g^{(0)}_{\Delta,\ell}(V)=(1-V)^{\ell}{}_2F_1\left(\frac{\Delta+\ell}{2},\frac{\Delta+\ell}{2},\Delta+\ell,1-V\right)\ . \tag{5.41}$$

Similarly, the vector contributes $\frac{\pi^3}{4}(\sigma-\tau)U^{\frac{1}{2}}\lambda_s g^{\rm coll}_{2,1}(V)$ and the graviton contributes $-\frac{1}{256}\left(3\pi^3\right)U^{\frac{1}{2}}\lambda_s g^{\rm coll}_{3,2}(V)$. Taking into account the normalization difference, $g^{\rm there}=\frac{(\epsilon)_\ell}{4^\Delta(2\epsilon)_\ell}g^{\rm here}$, we find the ratio $1:-1:\frac{1}{4}$ in (C.1-3) of [90].

[5]Note that we extract from all terms a common Gamma factor $\Gamma^2[2-s/2]\Gamma^2[2-t/2]\Gamma^2[2-u/2]$.

we get

$$\mathcal{M}_{\text{contact}} = \frac{\pi\lambda_s}{2}\left(-s - u\sigma^2 - t\tau^2 + 4(t+u)\sigma\tau + 4(s+u)\sigma + 4(s+t)\tau\right) \ . \tag{5.42}$$

Now let us fix the last coefficient λ_s. In [91] the three-point functions of the half-BPS operators were computed from supergravity

$$\langle \mathcal{O}_2^{I_1}\mathcal{O}_2^{I_2}\mathcal{O}_2^{I_3}\rangle = \frac{2^{1/4}\sqrt{3\pi}}{N^{3/4}} \times \frac{\langle C^{I_1}C^{I_2}C^{I_3}\rangle}{x_{12}^2 x_{13}^2 x_{23}^2} \tag{5.43}$$

where $\langle C^{I_1}C^{I_2}C^{I_3}\rangle$ is some R-symmetry tensor structure. In the small U and small V limit, the s-channel exchange of operator $\mathcal{O}_2$ contributes to the four-point function by (see Appendix B of [5] for relating the C-symbols to the R-symmetry polynomials)

$$\frac{1}{2}\left(\frac{2^{1/4}\sqrt{3\pi}}{N^{3/4}}\right)^2 (\sigma + \tau - \frac{1}{4})\, U^{1/2} g_{1,0}^{\text{coll}}(V) + \dots \ . \tag{5.44}$$

On the other hand, closing the contours in the inverse Mellin representation gives the following leading contribution in the $(\sigma+\tau-\frac{1}{4})$ R-symmetry channel

$$-2\pi^3\lambda_s\, (\sigma + \tau - \frac{1}{4}) \times U^{1/2} g_{1,0}^{\text{coll}}(V) + \dots \ . \tag{5.45}$$

Matching these two expressions gives

$$\lambda_s = -\frac{3\sqrt{2}}{4\pi^2 N^{3/2}} \ . \tag{5.46}$$

Analysis of Anomalous Dimension

In [92, 93, 94], it was observed that supergravity duals saturate the conformal bootstrap bounds for CFTs with maximal supersymmetry in four and six dimensions. A curious question therefore is if the same phenomenon persists also in three dimensions. In this subsection, we extract CFT data from the Mellin amplitude. We focus here on the anomalous dimension of the R-symmetry singlet scalar double-trace operators $[\mathcal{O}_2\mathcal{O}_2]_{\Delta=2,\ell=0,\text{singlet}}$. We compare the analytic result with the bound obtained from the numerical bootstrap. We find that the bootstrap estimation of the bound on the $1/C_T$ slope at large C_T is reasonably close to the supergravity prediction.

Let us start with the leading disconnect piece of the four-point correlator

$$\mathcal{G}_{\rm disc} = 1 + \sigma^2 U + \frac{\tau^2 U}{V} \ . \tag{5.47}$$

Decomposing this correlator into different R-symmetry channels requires the use of the following $SO(8)$ R-symmetry polynomials which form a basis of degree-two polynomials

$$\begin{aligned}
\mathbf{1}: &\qquad 1\ ,\\
\mathbf{28}: &\qquad \sigma-\tau\ ,\\
\mathbf{35}_c: &\qquad \sigma+\tau-\tfrac{1}{4}\ ,\\
\mathbf{300}: &\qquad \sigma^2-2\sigma\tau+\tau^2-\tfrac{1}{3}\sigma-\tfrac{1}{3}\tau+\tfrac{1}{21}\ ,\\
\mathbf{567}_c: &\qquad \sigma^2-\tau^2-\tfrac{2}{5}\sigma+\tfrac{2}{5}\tau\ ,\\
\mathbf{294}_c: &\qquad \sigma^2+4\sigma\tau-\tfrac{2\sigma}{3}+\tau^2-\tfrac{2\tau}{3}+\tfrac{1}{15}\ .
\end{aligned} \tag{5.48}$$

The projection of the disconnect correlator onto the R-symmetry singlet channel gives

$$\mathcal{G}_{\text{disc, singlet}} = 1 + \frac{U(1+V)}{35V} \ . \tag{5.49}$$

The first term of this singlet sector correlator is due to the exchange of the identity operator. The second term admits a decomposition into conformal blocks of double-trace operators $[\mathcal{O}_2\mathcal{O}_2]_{\Delta=2+2n+\ell,\ell,\text{singlet}}$ of the schematic form $:\mathcal{O}_2^{IJ}\Box^n\partial^\ell\mathcal{O}_2^{IJ}:$ with ℓ even and their R-symmetry singlet superconformal descendants. For double-trace operators with $\Delta=2$, $\ell=0$, there is one *unique* such operator, namely, $[\mathcal{O}_2\mathcal{O}_2]_{\Delta=2,\ell=0,\text{singlet}}$. We can easily extract its zeroth order squared OPE coefficient from the above disconnected correlator and we get

$$a^{(0)}_{n=0,\ell=0} \;=\; \frac{2}{35} \ . \tag{5.50}$$

In the small U and small V limit, the anomalous dimension $\gamma^{(1)}_{n=0,\ell=0}$ of this double-trace operator appears inside the term proportional to $U\log U$ of the four-point function

$$\begin{aligned}
\mathcal{G}_{\rm singlet}(U,V) &= A(V)U\log(U)+\dots\ ,\\
A(V) &= \frac{1}{2}a^{(0)}_{n=0,\ell=0}\gamma^{(1)}_{n=0,\ell=0}g^{\rm coll}_{2,0}(V) + \sum_{\ell\geq 2,\text{ even}}^{\infty}\sum_i \frac{1}{2}a^{(0)}_{n=0,\ell;i}\gamma^{(1)}_{n=0,\ell;i}g^{\rm coll}_{2+\ell,\ell}(V)\ .
\end{aligned} \tag{5.51}$$

The $U\log(U)$ term comes from the residue at the double pole $s=2$ in the inverse Mellin transformation, and the function $A(V)$ is further obtained by closing the contour of t. To extract the $\ell=0$ contribution, we use the following orthogonality property of ${}_2F_1$ [95],

$$
\begin{aligned}
&F_a(z) \equiv {}_2F_1(a,a;2a;z)\ ,\\
&\oint_{z=0} \frac{dz}{2\pi i} z^{m-m'-1} F_{\Delta+m}(z) F_{1-\Delta-m'}(z) = \delta_{m,m'}\ .
\end{aligned}
\tag{5.52}
$$

Evaluating the integrals, we find that the contact part of the Mellin amplitude contributes to $a^{(0)}_{n=0,\ell=0}\gamma^{(1)}_{n=0,\ell=0}$ by $-\frac{36\pi}{35}\lambda_s$ and the exchange part contributes to it by $\frac{106\pi}{35}\lambda_s$. The total anomalous dimension therefore is

$$
\gamma^{(0)}_{n=0,\ell=0} = 35\pi\lambda_s \;\; = \;\; -\frac{1120}{\pi^2}\frac{1}{C_T} \approx -113.5\frac{1}{C_T} \tag{5.53}
$$

where $C_T = \frac{64\sqrt{2}}{3\pi}N^{3/2}$ in the convention of [90]. Supergravity hence yields the following large C_T expansion for the conformal dimensions of the double-trace operators

$$
\Delta_0 \;\; \approx \;\; 2 - 113.5\frac{1}{C_T} + \ldots\ . \tag{5.54}
$$

In [90], a numerical upper bound for the spin zero operator was reported, $\Delta_0^* \gtrsim 2.03 - 94.6/C_T + \ldots$. We see that this bound is compatible with the conjecture that supergravity should saturate the bootstrap bound, although the difference is still quite significant. This discrepancy can be partially explained by the slow convergence in the numerics for the scalar sector.[6] Indeed, with more computational power a better estimation of the bound is [96]

$$
\Delta_0^* \gtrsim 2.01 - 104/C_T + \ldots\ , \tag{5.55}
$$

and shows improved agreement with the supergravity result.

Here for simplicity, we computed only the scalar singlet double-trace operator with the lowest conformal dimension. A more systematic analysis that extracts the low-lying CFT data from our amplitude was recently performed in [97]. It was found that for half and quarter-BPS operators, the correction to the OPE coefficients agree precisely with the localization result [96]. OPE coefficients for other BPS and anomalous dimensions for non-BPS operators were compared with the 3d $\mathcal{N}=8$ bootstrap results at large central charge and matched nicely with the numerics.

[6] The same phenomenon was observed in [92, 93].

Chapter 6

Conclusions and Open Questions

Computing boundary correlation functions from AdS/CFT used to be a notoriously difficult task, with only a handful results obtained in the literature. These holographic correlators, despite being very important observables, remained very mysterious twenty years after the birth of AdS/CFT. In this thesis we started developing a deeper understanding of these objects. We presented three complementary approaches, which allow us to efficiently compute holographic correlators in a number of theories in different spacetime dimensions. Our novel results not only showed that holographic correlators are much simpler than previously understood, but also revealed intricate structures and principles in terms of which these correlators are organized. It should be apparent that there is a great deal more to learn about this subject. The investigations of this thesis open many interesting questions for future studies. We conclude with a list of natural next steps, some of which we are currently pursuing.

- The remarkable simplicity of the general formula (4.43) for all one-half BPS four-point functions in $\mathcal{N} = 4$ SYM is a welcome surprise. But does this succinct auxiliary object have an intrinsic physical meaning? It is also interesting to see how our results of four dimensional $\mathcal{N} = 4$ SYM at infinite 't Hooft coupling can be connected to the integrability program[1].

[1] See, *e.g.*, [98, 99, 100] for recent work on four-point functions.

- In Chapter 4, we reformulated the task of computing holographic four-point functions into solving an algebraic bootstrap problem. How do we generalize this approach to set up a problem for higher-point correlation functions?

- Alternatively, we may ask if there is a more constructive approach to reproduce our results, based on on-shell recursion relations (*à la* BCFW [101]). Such an approach if exists will lend itself more easily to the generalization to n-point functions.

- For correlators in the $AdS_4 \times S^7$ background, a curious question remains whether the Mellin amplitude also admits some hidden structure which allows them to be repackaged in terms of some simpler auxiliary amplitude. The answer to this question is not obvious from the position space solution to the superconformal Ward identity because we do not know how to interpret the non-local differential operators in Mellin space. In this dissertation we have only considered the lowest KK modes. The natural next step is to extend our results to four-point functions of massive KK modes, in particular, the next-next-to-extremal correlators. It might be possible to spot these structures directly from the Mellin amplitudes themselves.

- In this dissertation, we focused on models with maximal superconformal symmetry. The techniques from Chapter 5 can also be adapted to consider backgrounds with less superconformal symmetry, *e.g.*, eight Poincaré supercharges [7]. The computation of one-half BPS four-point correlators in such backgrounds was explored in [7] for two interesting models, namely, the Seiberg theories in five dimensions [102] and E-string theories in six dimensions [103, 104]. Both theories admit appropriate limit in which they can be approximated by bulk supergravity theories. We focused in [7] on the correlators involving only massless modes. It would be interesting to further study massive correlators. It would also be interesting to apply the techniques to other theories in other spacetime dimensions, *e.g.*, 4d $\mathcal{N} = 2$ theories.

Appendix A

Formulae for Exchange Witten Diagrams

We are interested in here the case where the exchange diagrams truncate to a finite number of D-functions, as a result of the conspiracy of the spectrum and the space-time dimension. A simple general method for calculating such exchange diagrams in AdS_{d+1} was found [36]. We collect in this Appendix the relevant formulae needed in the computation of four-point function of identical scalars. The external operators have conformal dimension Δ and the exchanged operator conformal dimension δ.

Scalar exchanges

$$S(x_1, x_2, x_3, x_4) = \sum_{k=\delta/2}^{\Delta-1} a_k |x_{12}|^{-2\Delta+2k} D_{k,k,\Delta,\Delta}\,, \tag{A.1}$$

where

$$a_{k-1} = \frac{(k-\frac{\delta}{2})(k-\frac{d}{2}+\frac{\delta}{2})}{(k-1)^2} a_k \tag{A.2}$$

and

$$a_{\Delta-1} = \frac{1}{4(\Delta-1)^2}\,. \tag{A.3}$$

Vector exchanges

$$V(x_1,x_2,x_3,x_4) = \sum_{k=k_{\min}}^{k_{\max}} |x_{12}|^{-2\Delta+2k} a_k \Delta \Big({x_{24}}^2 D_{k,k+1,\Delta,\Delta+1} \\ + {x_{13}}^2 D_{k+1,k,\Delta+1,\Delta} - {x_{23}}^2 D_{k,k+1\Delta+1,\Delta} - {x_{14}}^2 D_{k+1,k,\Delta,\Delta+1} \Big) , \tag{A.4}$$

where

$$\begin{aligned} k_{\min} &= \frac{d-2}{4} + \frac{1}{4}\sqrt{(d-2)^2 + 4(\delta-1)(\delta-d+1)} , \\ k_{\max} &= \Delta - 1 , \\ a_{k-1} &= \frac{2k(2k+2-d) - (\delta-1)(\delta-d+1)}{4(k-1)k} a_k , \\ a_{\Delta-1} &= \frac{1}{2(\Delta-1)} . \end{aligned} \tag{A.5}$$

Graviton exchanges

$$G(x_1,x_2,x_3,x_4) = \sum_{k=k_{\min}}^{k_{\max}} {x_{12}}^{-2\Delta+2k} a_k \Big((\Delta^2 + \frac{1}{d-1}\Delta(\Delta-d)) D_{k,k,\Delta,\Delta} \\ - 2\Delta^2 \big({x_{13}}^2 D_{k+1,k,\Delta+1,\Delta} + {x_{14}}^2 D_{k+1,k,\Delta,\Delta+1} \big) \\ + 4\Delta^2 {x_{13}}^2 {x_{14}}^2 D_{k+2,k,\Delta+1,\Delta+1} \Big) , \tag{A.6}$$

where

$$\begin{aligned} k_{\min} &= \frac{d}{2} - 1 , \\ k_{\max} &= \Delta - 1 , \\ a_{k-1} &= \frac{k+1-\frac{d}{2}}{k-1} a_k , \\ a_{\Delta-1} &= -\frac{\Delta}{2(\Delta-1)} . \end{aligned} \tag{A.7}$$

Massive symmetric tensor exchanges

The Witten diagrams for massive symmetric tensor exchange were worked out in [18] for the general case[1] of AdS_d, and applied to the AdS_5 case. We fixed a small error in [18], which only affects the results for $d \neq 5$ and thus leaves the conclusions of [18] unaltered. For future reference, we reproduce the general calculation here. Due to the complexity of the explicit form of the general solution, we will not present here the answer as a sum of D-functions. Instead we will break down the evaluation into a few parts and give the prescription of how to assemble them into a sum of D-functions.

The four-point amplitude $T(x_1, x_2, x_3, x_4)$ due to the exchange of a massive symmetric tensor of dimension δ is

$$T(x_1, x_2, x_3, x_4) = \int_{AdS} dw A_{\mu\nu}(w, x_1, x_2) T^{\mu\nu}(x_3, x_4, w) \tag{A.8}$$

where

$$\begin{aligned} T_{\mu\nu} = \partial_\mu K_\Delta(x_3) \partial_\nu K_\Delta(x_4) - \frac{g_{\mu\nu}}{2} \big(\partial_\rho K_\Delta(x_3) \partial_\rho K_\Delta(x_4) \big) \\ + \frac{g_{\mu\nu}}{4} \big((2\Delta(\Delta - d + 1) - f) K_\Delta(x_3) K_\Delta(x_4) \big). \end{aligned} \tag{A.9}$$

Here $f = \delta(\delta - d + 1)$ is the m^2 of the exchanged massive tensor and

$$K_n(x_i) = \left(\frac{w_0}{(w - x)^2} \right)^n \tag{A.10}$$

is the scalar bulk-to-boundary propagator. By conformal inversion and translation $A_{\mu\nu}$ can be rewritten as

$$A_{\mu\nu}(w, 0, x) = \frac{1}{x^{2\Delta} w^4} J_{\mu\lambda} J_{\nu\rho} I_{\lambda\rho}(w' - x') \ , \tag{A.11}$$

with $w'_\mu = \frac{w_\mu}{w^2}$, $x'_\mu = \frac{x_\mu}{x^2}$. The ansatz is

$$I_{\mu\nu}(w) = g_{\mu\nu} h(t) + P_\mu P_\nu \phi(t) + \nabla_\mu \nabla_\nu X(t) + \nabla_{(\mu} P_{\nu)} Y(t) \, . \tag{A.12}$$

[1] For easier comparison with the equations of [18], **in this subsection** we change our conventions such that d **is the bulk dimension**. In this subsection and in this subsection only, we are working in AdS_d rather than in AdS_{d+1}.

For any scalar function $b(t)$,

$$\begin{aligned}\nabla_\mu \nabla_\nu b(t) = &\frac{2w_\mu w_\nu}{w^4} t b'(t) + 6(P_\mu - \frac{w_\mu}{w^2})(P_\nu - \frac{w_\nu}{w^2}) t b'(t) \\ &+ 4(P_\mu - \frac{w_\mu}{w^2})(P_\nu - \frac{w_\nu}{w^2}) t^2 b''(t) - 2g_{\mu\nu} t b'(t) \ ,\end{aligned} \tag{A.13}$$

$$\nabla_{(\mu} P_{\nu)} b(t) = 2(P_\mu P_\nu - g_{\mu\nu}) b(t) + 2(2P_\mu P_\nu - \frac{P_\mu w_\nu + P_\nu w_\mu}{w^2}) t b'(t) \, . \tag{A.14}$$

Here, as standard in the literature, we have denoted

$$P_\mu = \frac{\delta_{0\mu}}{w_0} \ , \quad t = \frac{(w_0)^2}{w^2} \ . \tag{A.15}$$

The functions $h(t)$, $\phi(t)$, $X(t)$, $Y(t)$ are subject to the following set of equations,

$$h(t) = -\frac{1}{d-2}\phi(t) + \frac{f}{d-2} X(t) \ , \tag{A.16}$$

$$Y(t) = a + \frac{1}{2f}(4t(t-1)\phi'(t) + (2d-6)\phi(t) + 2\Delta t^\Delta) \ , \tag{A.17}$$

$$\begin{aligned}X(t) = &\frac{1}{2(d-1)f(d+f-2)}\Bigg(2a(d-2)(2d-3)f + \big[2(d-3)(d-2)^2 + df\big]\phi(t) \\ &+ (d-2)\Big[t^\Delta(-f + 2(-\Delta^2 + \Delta(d-2) + 2\Delta^2 t)) + 2t(2t+d-3)(4t-3)\phi'(t) \\ &+ 4t^2(t-1)(2t-1)\phi''(t)\Big]\Bigg) \ ,\end{aligned} \tag{A.18}$$

$$4t^2(t-1)\phi''(t) + (12t^2 + (2d-14)t)\phi'(t) + (f+2d-6)\phi(t) + 2fa + 2\Delta(\Delta+1)t^\Delta = 0 \ , \tag{A.19}$$

where a is an integration constant that will cancel out when we substitute the solution into the ansatz for $I_{\mu\nu}$. These equations come from the action of the modified Ricci operator $W_{\mu\nu}{}^{\rho\lambda}$[2] on $A_{\mu\nu}$ and equating terms of the same structure. We omitted the tedious algebra here.

We start from the last equation and look for a polynomial solution for $\phi(t)$. As we will see shortly, a polynomial solution will lead to a truncation of the exchange diagram to finitely many D-functions. We find

$$
\begin{aligned}
\phi(t) &= -\frac{2a\delta(\delta-d+1)}{(\delta-2)(\delta-d+3)} + \sum_{k=k_{\min}}^{k_{\max}} a_k t^k \ , \\
k_{\min} &= \frac{\delta-2}{2} \ , \\
k_{\max} &= \Delta - 1 \ , \\
a_{k-1} &= \frac{(k+\frac{3-d+\delta}{2})(1+k-\frac{\delta}{2})}{(k-1)(k+1)} a_k \ , \\
a_{\Delta-1} &= -\frac{\Delta}{2\Delta-1} \ .
\end{aligned}
\tag{A.21}
$$

For the polynomial solution to exist, $k_{\max} - k_{\min} = \Delta - \delta/2$ must be an non-negative integer. When $2\Delta = \delta - 2$, which is the extremal case, we see the polynomial solution will stop from existing.

After obtaining the polynomial solution for $\phi(t)$, we can easily solve out $h(t)$, $X(t)$, $Y(t)$ from the rest three equations. And it is easy to see $I_{\mu\nu}(t)$ contains only finitely many terms of the following four types

$$
g_{\mu\nu} t^n, \quad \frac{P_\mu w_\nu}{w^2} t^n, \quad \frac{w_\mu w_\nu}{w^4} t^n, \quad P_\mu P_\nu t^n \ . \tag{A.22}
$$

We can get $A_{\mu\nu}$ from $I_{\mu\nu}(w'-x')$ with the following substitutions:

$$
\begin{aligned}
&P'_\nu \frac{J_{\mu\nu}}{w^2} \to R_\mu \equiv P_\mu - 2\frac{(w-x_1)_\mu}{(w-x_1)^2}, \\
&\frac{J_{\mu\nu}}{w^2} \frac{(w'-x')_\mu}{(w'-x')^2} \to Q_\mu \equiv -\frac{(w-x_1)_\mu}{(w-x_1)^2} + \frac{(w-x_2)_\mu}{(w-x_2)^2} \ , \\
&\frac{J_{\mu\rho}}{w^2} g'_{\rho\lambda} \frac{J_{\lambda\nu}}{w^2} \to g_{\mu\nu}, \qquad t'^n \to x_{12}^{2n} K_n(x_1) K_n(x_2) \ .
\end{aligned}
\tag{A.23}
$$

[2] There is an error in (E.4) of [18] that must be fixed in order to generalize to arbitrary d. The correct equation is [105]

$$
W_{\mu\nu}{}^{\rho\lambda}\phi_{\rho\lambda} = -\nabla_\rho\nabla^\rho\phi_{\mu\nu} + \nabla_\nu\nabla^\rho\phi_{\rho\mu} + \nabla_\nu\nabla^\rho\phi_{\rho\nu} - \nabla_\mu\nabla^\nu\phi^\rho_\rho - ((2-f)\phi_{\mu\nu} + \frac{2d-4+f}{2-d} g_{\mu\nu}\phi^\rho_\rho) \tag{A.20}
$$

The last step is to contract $A_{\mu\nu}$ with $T_{\mu\nu}$. We list below the following handy contraction formulae,

$$
\begin{aligned}
&Q_\mu Q_\mu = x_{12}^2 K_1(x_1) K_1(x_2) \ , \\
&Q_\mu \partial_\mu K_\Delta(x_i) = \Delta K_{\Delta+1}(x_i)(-x_{1i}^2 K_1(x_1) + x_{2i}^2 K_1(x_2)) \ , \\
&R_\mu R_\mu = 1 \ , \\
&R_\mu \partial_\mu K_\Delta(x_i) = \Delta(K_\Delta(x_i) - 2x_{1i}^2 K_1(x_1) K_{\Delta+1}(x_i)) \ , \\
&R_\mu Q_\mu = x_{12}^2 K_1(x_1) K_1(x_2) \ .
\end{aligned}
\tag{A.24}
$$

The above derivation amounts to an algorithm to write the requisite exchange diagrams as a sum of D-function. The explicit final result is too long to be reproduced here.

Appendix B

Simplification in the Contact Diagrams

In this Appendix we show that the zero-derivative contact vertex can be absorbed into the two-derivative ones when the dimension of external scalar particle does not equal the spacetime dimension of the boundary theory.

A zero-derivative contact vertex takes the form of

$$V_{0-\partial} = C_{\alpha_1\alpha_2\alpha_3\alpha_4} \int_{AdS_{d+1}} dX s^{\alpha_1}(X) s^{\alpha_2}(X) s^{\alpha_3}(X) s^{\alpha_4}(X)\,, \tag{B.1}$$

while a two-derivative contact vertex is

$$V_{2-\partial} = S_{\alpha_1\alpha_2\alpha_3\alpha_4} \int_{AdS_{d+1}} dX \nabla s^{\alpha_1}(X) \nabla s^{\alpha_2}(X) s^{\alpha_3}(X) s^{\alpha_4}(X)\,. \tag{B.2}$$

Here α_i collectively denotes the R-symmetry index of ith field s.

Following the standard procedure in AdS supergravity calculation, we substitute in the on-shell value of scalar field

$$s^\alpha(X) = \int_{\mathbb{R}^d} dP K_\Delta(X,P) s^\alpha(P) \tag{B.3}$$

so that it is determined by its boundary value $s^\alpha(P)$. Then the two types of contact vertices become

$$\begin{aligned} V_{0-\partial} = C_{\alpha_1\alpha_2\alpha_3\alpha_4} &\int_{AdS_{d+1}} dX \int_{\mathbb{R}^d} \prod dP_i \\ &\times K_\Delta(X,P_1) K_\Delta(X,P_2) K_\Delta(X,P_3) K_\Delta(X,P_4) s^{\alpha_1}(P_1) s^{\alpha_2}(P_2) s^{\alpha_3}(P_3) s^{\alpha_4}(P_4)\,, \end{aligned} \tag{B.4}$$

$$V_{2-\partial} = S_{\alpha_1\alpha_2\alpha_3\alpha_4} \int_{AdS_{d+1}} dX \int_{\mathbb{R}^d} \prod dP_i \; s^{\alpha_1}(P_1) s^{\alpha_2}(P_2) s^{\alpha_3}(P_3) s^{\alpha_4}(P_4) \\ \times \nabla K_\Delta(X, P_1) \nabla K_\Delta(X, P_2) K_\Delta(X, P_3) K_\Delta(X, P_4) \; . \tag{B.5}$$

Because the external fields are identical, $C_{\alpha_1\alpha_2\alpha_3\alpha_4}$ is totally symmetric while $S_{\alpha_1\alpha_2\alpha_3\alpha_4}$ is only required to be symmetric under $\alpha_1 \leftrightarrow \alpha_2$, $\alpha_3 \leftrightarrow \alpha_4$ and $(\alpha_1\alpha_2) \leftrightarrow (\alpha_3\alpha_4)$. This in particular means that the totally symmetric $C_{\alpha_1\alpha_2\alpha_3\alpha_4}$ can be a $S_{\alpha_1\alpha_2\alpha_3\alpha_4}$. Let us see what the consequence is if we take $S_{\alpha_1\alpha_2\alpha_3\alpha_4} = C_{\alpha_1\alpha_2\alpha_3\alpha_4}$,

$$\begin{aligned} V_{2-\partial} &= C_{\alpha_1\alpha_2\alpha_3\alpha_4} \int dX \int \prod dP_i \nabla K_1 \nabla K_2 K_3 K_4 s^{\alpha_1} s^{\alpha_2} s^{\alpha_3} s^{\alpha_4} \\ &= \frac{1}{6} C_{\alpha_1\alpha_2\alpha_3\alpha_4} \int dX \int \prod dP_i \; s^{\alpha_1} s^{\alpha_2} s^{\alpha_3} s^{\alpha_4} \\ &\quad \times (\nabla K_1 \nabla K_2 K_3 K_4 + \nabla K_1 K_2 \nabla K_3 K_4 + \nabla K_1 K_2 K_3 \nabla K_4 \\ &\quad + K_1 \nabla K_2 \nabla K_3 K_4 + K_1 \nabla K_2 K_3 \nabla K_4 + K_1 K_2 \nabla K_3 \nabla K_4) \end{aligned} \tag{B.6}$$

Here $K_i \equiv K_\Delta(P_i)$ and we have used the total symmetry of $C_{\alpha_1\alpha_2\alpha_3\alpha_4}$ to symmetrize the expression. If we now perform the AdS integral first, each term can be written as a sum of D-functions. For example

$$\int_{AdS_{d+1}} dX \nabla K_1 \nabla K_2 K_3 K_4 = \Delta^2 (D_{\Delta,\Delta,\Delta,\Delta} - 2{x_{12}}^2 D_{\Delta+1,\Delta+1,\Delta,\Delta}) \; . \tag{B.7}$$

The two-derivative vertex then becomes

$$\begin{aligned} V_{2-\partial} = \frac{\Delta^2}{6} C_{\alpha_1\alpha_2\alpha_3\alpha_4} \int \prod dP_i s^{\alpha_1} s^{\alpha_2} s^{\alpha_3} s^{\alpha_4} \times (& 6 D_{\Delta,\Delta,\Delta,\Delta} - 2{x_{12}}^2 D_{\Delta+1,\Delta+1,\Delta,\Delta} \\ & - 2{x_{13}}^2 D_{\Delta+1,\Delta,\Delta+1,\Delta} - 2{x_{14}}^2 D_{\Delta+1,\Delta,\Delta,\Delta+1} - 2{x_{23}}^2 D_{\Delta,\Delta+1,\Delta+1,\Delta} \\ & - 2{x_{24}}^2 D_{\Delta,\Delta+1,\Delta,\Delta+1} - 2{x_{34}}^2 D_{\Delta,\Delta,\Delta+1,\Delta+1}) \; . \end{aligned} \tag{B.8}$$

Using the identity

$$\frac{(2\Delta - d/2)}{\Delta} D_{\Delta,\Delta,\Delta,\Delta} = x_{14}^2 D_{\Delta+1,\Delta,\Delta,\Delta+1} + x_{24}^2 D_{\Delta,\Delta+1,\Delta,\Delta+1} + x_{34}^2 D_{\Delta,\Delta,\Delta+1,\Delta+1} \; , \tag{B.9}$$

we simplify the expression to

$$
\begin{aligned}
V_{2-\partial} &= \frac{\Delta^2}{6} C_{\alpha_1\alpha_2\alpha_3\alpha_4} \int \prod dP_i s^{\alpha_1} s^{\alpha_2} s^{\alpha_3} s^{\alpha_4} (6D_{\Delta,\Delta,\Delta,\Delta} - \frac{2(4\Delta - d)}{\Delta} D_{\Delta,\Delta,\Delta,\Delta}) \\
&= \frac{\Delta(d-\Delta)}{3} C_{\alpha_1\alpha_2\alpha_3\alpha_4} \int \prod dP_i s^{\alpha_1} s^{\alpha_2} s^{\alpha_3} s^{\alpha_4} D_{\Delta,\Delta,\Delta,\Delta} \\
&= \frac{\Delta(d-\Delta)}{3} V_{0-\partial} \, .
\end{aligned}
\tag{B.10}
$$

We have therefore proved that when $\Delta \neq d$, we can absorb the contribution from zero-derivative contact vertices into the two-derivative ones.

Appendix C

$k = 2$: A Check of the Domain-Pinching Mechanism

We computed the $k = 2$ correlator from supergravity, using the position space method of Section 2.3. We found

$$
\begin{aligned}
\mathcal{G}^{\text{sugra}}_{\text{conn}}(U,V;\sigma,\tau) = -\frac{2U}{N^2V} \times \Big(& -\sigma\tau\bar{D}_{3,2,1,2}U^2 + \tau\bar{D}_{3,2,1,2}U^2 + V\sigma\bar{D}_{3,2,2,1}U^2 \\
& -V\sigma\tau\bar{D}_{3,2,2,1}U^2 + V\sigma^2\bar{D}_{3,3,2,2}U^2 + V\tau^2\bar{D}_{3,3,2,2}U^2 + V\bar{D}_{3,3,2,2}U^2 - 4V\sigma\bar{D}_{3,3,2,2}U^2 \\
& -4V\tau\bar{D}_{3,3,2,2}U^2 - 2V\sigma\tau\bar{D}_{3,3,2,2}U^2 + 2\tau^2\bar{D}_{2,1,1,2}U - 2\sigma\tau\bar{D}_{2,1,1,2}U - 2\tau\bar{D}_{2,1,1,2}U \\
& +2V\sigma^2\bar{D}_{2,1,2,1}U - 2V\sigma\bar{D}_{2,1,2,1}U - 2V\sigma\tau\bar{D}_{2,1,2,1}U - 2\tau^2\bar{D}_{2,1,2,3}U - \sigma\tau\bar{D}_{2,1,2,3}U \\
& +\tau\bar{D}_{2,1,2,3}U - 2V\sigma^2\bar{D}_{2,1,3,2}U + V\sigma\bar{D}_{2,1,3,2}U - V\sigma\tau\bar{D}_{2,1,3,2}U + \sigma\tau\bar{D}_{2,2,1,3}U \\
& -\tau\bar{D}_{2,2,1,3}U - 6V\sigma^2\bar{D}_{2,2,2,2}U - 6V\tau^2\bar{D}_{2,2,2,2}U - 6V\bar{D}_{2,2,2,2}U + 20V\sigma\bar{D}_{2,2,2,2}U \\
& +20V\tau\bar{D}_{2,2,2,2}U + 20V\sigma\tau\bar{D}_{2,2,2,2}U - V^2\sigma\bar{D}_{2,2,3,1}U + V^2\sigma\tau\bar{D}_{2,2,3,1}U \\
& +V\sigma^2\bar{D}_{2,2,3,3}U + V\tau^2\bar{D}_{2,2,3,3}U + V\bar{D}_{2,2,3,3}U - 4V\sigma\bar{D}_{2,2,3,3}U - 4V\tau\bar{D}_{2,2,3,3}U \\
& -2V\sigma\tau\bar{D}_{2,2,3,3}U + V\sigma^2\bar{D}_{2,3,2,3}U + V\tau^2\bar{D}_{2,3,2,3}U + V\bar{D}_{2,3,2,3}U - 4V\sigma\bar{D}_{2,3,2,3}U \\
& -2V\tau\bar{D}_{2,3,2,3}U - 4V\sigma\tau\bar{D}_{2,3,2,3}U + V^2\bar{D}_{2,3,3,2}U + V^2\sigma^2\bar{D}_{2,3,3,2}U + V^2\tau^2\bar{D}_{2,3,3,2}U \\
& -2V^2\sigma\bar{D}_{2,3,3,2}U - 4V^2\tau\bar{D}_{2,3,3,2}U - 4V^2\sigma\tau\bar{D}_{2,3,3,2}U - 2V\sigma^2\bar{D}_{3,1,2,2}U \\
& -2\tau^2\bar{D}_{3,1,2,2}U - V\sigma\bar{D}_{3,1,2,2}U + V\sigma\tau\bar{D}_{3,1,2,2}U + \sigma\tau\bar{D}_{3,1,2,2}U \\
& -\tau\bar{D}_{3,1,2,2}U + 2V\sigma^2\bar{D}_{3,1,3,3}U + 2\tau^2\bar{D}_{3,1,3,3}U + V\sigma^2\bar{D}_{3,2,2,3}U + V\tau^2\bar{D}_{3,2,2,3}U \\
& +V\bar{D}_{3,2,2,3}U - 2V\sigma\bar{D}_{3,2,2,3}U - 4V\tau\bar{D}_{3,2,2,3}U - 4V\sigma\tau\bar{D}_{3,2,2,3}U + V\sigma^2\bar{D}_{3,2,3,2}U \\
& +V\tau^2\bar{D}_{3,2,3,2}U + V\bar{D}_{3,2,3,2}U - 4V\sigma\bar{D}_{3,2,3,2}U - 2V\tau\bar{D}_{3,2,3,2}U - 4V\sigma\tau\bar{D}_{3,2,3,2}U \\
& +2V\bar{D}_{1,1,2,2} - 2V\sigma\bar{D}_{1,1,2,2} - 2V\tau\bar{D}_{1,1,2,2} + V\sigma\bar{D}_{1,2,2,3} - V\tau\bar{D}_{1,2,2,3} \\
& -V^2\sigma\bar{D}_{1,2,3,2} + V^2\tau\bar{D}_{1,2,3,2} - 2V\bar{D}_{2,1,2,3} - V\sigma\bar{D}_{2,1,2,3} + V\tau\bar{D}_{2,1,2,3} \\
& -2V\bar{D}_{2,1,3,2} + V\sigma\bar{D}_{2,1,3,2} - V\tau\bar{D}_{2,1,3,2} + 2V\bar{D}_{3,1,3,3}\Big) \,.
\end{aligned}
\tag{C.1}
$$

We can get the Mellin transform of $\mathcal{G}_{\text{sugra,conn}}(U,V;\sigma,\tau)$ by Mellin-transforming each $\bar{D}$-function in $\mathcal{G}_{\text{sugra,conn}}$. Formally, the transformation reads

$$
M(s,t;\sigma,\tau) = \int_0^\infty dU dV U^{-s/2-1} V^{-t/2+2-1} \mathcal{G}_{\text{sugra,conn}}(U,V;\sigma,\tau) \,, \tag{C.2}
$$

but notice each $\bar{D}$-function may come with a different fundamental domain of s and t in which the integrals converge. These fundamental domains are defined by the positivity condition of the Gamma function arguments. Although no ambiguity arises when analytically continue the Mellin transformation outside this domain due to the absence of branch cuts, it is imperative to have the knowledge of the fundamental domain as the contour needs to be placed inside the fundamental domain in order to reproduce precisely the $\bar{D}$-function via the inverse Mellin-transformation. To keep track of this information, in the following expression we simply keep the Gamma functions from each $\bar{D}$-function, and the domain information can be extracted by requiring that the arguments of the Gamma functions have positive real part. With this proviso, the reduced Mellin amplitude reads

$$
\begin{aligned}
M(s,t;\sigma,\tau) = {}& \frac{4}{N^2}\Gamma[2-\frac{s}{2}]\Gamma[2-\frac{t}{2}]\Gamma[\frac{1}{2}(s+t-4)] \\
&\times\Big\{\Gamma\Big(2-\frac{s}{2}\Big)\times[\Gamma\Big(3-\frac{t}{2}\Big)\times(\sigma(\sigma-\tau+1)\Gamma\Big(\frac{1}{2}(s+t-6)\Big) \\
&-\Big(\sigma^2-2\sigma(2\tau+1)+\tau^2-4\tau+1\Big)\Gamma\Big(\frac{1}{2}(s+t-4)\Big)) \\
&+\Gamma\Big(2-\frac{t}{2}\Big)\times(\Big(-3\sigma^2+10\sigma(\tau+1)-3\tau^2+10\tau-3\Big)\Gamma\Big(\frac{1}{2}(s+t-4)\Big) \\
&+\Big(\sigma^2-4\sigma(\tau+1)+(\tau-1)^2\Big)\Gamma\Big(\frac{1}{2}(s+t-2)\Big)+\sigma(\sigma-\tau-1)\Gamma\Big(\frac{1}{2}(s+t-6)\Big)\Big) \\
&+\tau\Gamma\Big(1-\frac{t}{2}\Big)\times((\sigma-\tau+1)\Gamma\Big(\frac{1}{2}(s+t-4)\Big)+(-\sigma+\tau+1)\Gamma\Big(\frac{1}{2}(s+t-2)\Big))] \quad . \\
&+\Gamma\Big(3-\frac{s}{2}\Big)\times[\Gamma\Big(2-\frac{t}{2}\Big)\times(\sigma(\sigma+\tau-1)\Gamma\Big(\frac{1}{2}(s+t-6)\Big) \\
&-\Big(\sigma^2-2\sigma(\tau+2)+\tau^2-4\tau+1\Big)\Gamma\Big(\frac{1}{2}(s+t-4)\Big))-\sigma^2\Gamma\Big(3-\frac{t}{2}\Big)\Gamma\Big(\frac{1}{2}(s+t-6)\Big) \\
&+\tau\Gamma\Big(1-\frac{t}{2}\Big)\times((\sigma+\tau-1)\Gamma\Big(\frac{1}{2}(s+t-4)\Big)-\tau\Gamma\Big(\frac{1}{2}(s+t-2)\Big))] \\
&+\Gamma\Big(1-\frac{s}{2}\Big)\times[\Gamma\Big(3-\frac{t}{2}\Big)\times((\sigma-\tau+1)\Gamma\Big(\frac{1}{2}(s+t-4)\Big)-\Gamma\Big(\frac{1}{2}(s+t-2)\Big)) \\
&+\Gamma\Big(2-\frac{t}{2}\Big)\times((\sigma+\tau-1)\Gamma\Big(\frac{1}{2}(s+t-4)\Big)+(-\sigma+\tau+1)\Gamma\Big(\frac{1}{2}(s+t-2)\Big))]\Big\}.
\end{aligned}
\tag{C.3}
$$

We can use this example to illustrate how the free field correlator $\mathcal{G}_{\text{free,conn}}$ arises when one takes the inverse Mellin transform by the "contour pinching mechanism" described in Section 4.1.3. We will compare the expression that arises directly from the explicit supergravity calculation and the expression in the split form (4.1), both written as inverse Mellin transformations. Each summand in (C.3) contains a common Gamma function factor $\Gamma[2-\frac{s}{2}]\Gamma[2-\frac{t}{2}]\Gamma[\frac{s+t-4}{2}]$ which sets common bounds for the boundaries of all the fundamental domains – the real parts of s and t must be inside the big black-framed triangle in Figure. C.1. A closer look shows that some the summands in (C.3) have smaller domains. Imposing positivity of the rest of the Gamma functions in each term shows that there are four types of domains: the red $\{(2,4),(4,2),(4,4)\}$, green $\{(2,2),(4,0),(4,2)\}$ and orange $\{(0,4),(2,2),(2,4)\}$ triangles of size two (where by size we mean the length of its projection onto the $\Re(s)$ axis or $\Re(t)$ axis) and the bigger grey triangle $\{(0,4),(4,0),(4,4)\}$ of size four.

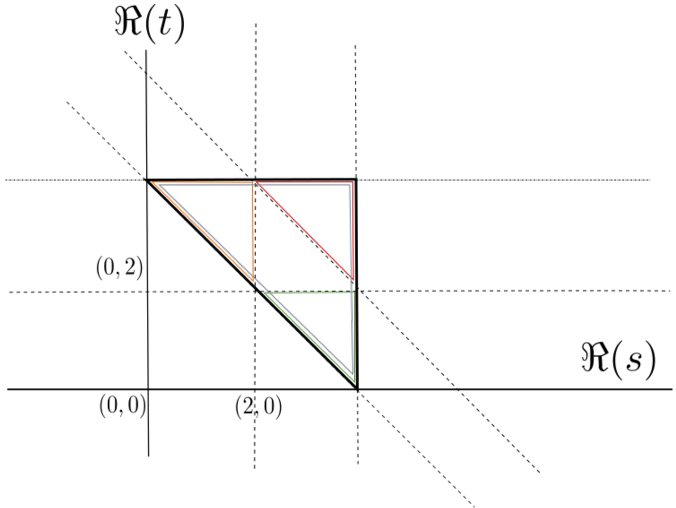

Figure C.1: The fundamental domains for the "unmassaged" supergravity result.

Now we take a look at the other form of the result where it has been split into two parts,

$$\mathcal{G}_{\text{conn}} = \mathcal{G}_{\text{free,conn}} + R\mathcal{H} \, . \tag{C.4}$$

The factor R was introduced before and we repeat here for reader's convenience,

$$R = \tau 1 + (1-\sigma-\tau)V + (-\tau-\sigma\tau+\tau^2)U + (\sigma^2-\sigma-\sigma\tau)UV + \sigma V^2 + \sigma\tau U^2 \, . \tag{C.5}$$

The first term $\mathcal{G}_{\text{free,conn}}$ is the connected free field four-point function, which can be computed by Wick contractions,

$$\mathcal{G}_{\text{free,conn}} = \frac{4}{N^2}\frac{U}{V}\left(\tau + V\sigma + U\sigma\tau\right) . \tag{C.6}$$

The function $\mathcal{H}$ was obtained in [106],

$$\mathcal{H} = -\frac{4}{N^2}U^2\bar{D}_{2422} \, . \tag{C.7}$$

We write $\mathcal{H}$ as an inverse Mellin transform,

$$\mathcal{H} = -\frac{4}{N^2}\times\frac{1}{4}\int_{\mathcal{C}} dsdt U^{s/2}V^{t/2-2}\Gamma[2-\frac{s}{2}]\Gamma[1-\frac{s}{2}]\Gamma[2-\frac{t}{2}]\Gamma[1-\frac{t}{2}]\Gamma[\frac{s+t}{2}-1]\Gamma[\frac{s+t}{2}]\,, \tag{C.8}$$

where $\mathcal{C}$ is associated with a point inside the fundamental domain

$$(s_0,t_0)\in\mathcal{D}=\{(s_0,t_0)|\Re(s)<2,\Re(t)<2,\Re(s)+\Re(t)>2\}\,, \tag{C.9}$$

represented by the yellow size-two triangle in Figure C.2. When multiplied by R, this domain will lead to six different domains generated by the six different shifts in R, namely, 1, U, V, UV, U^2, V^2. They are the six colored triangles[1] in Figure C.2.

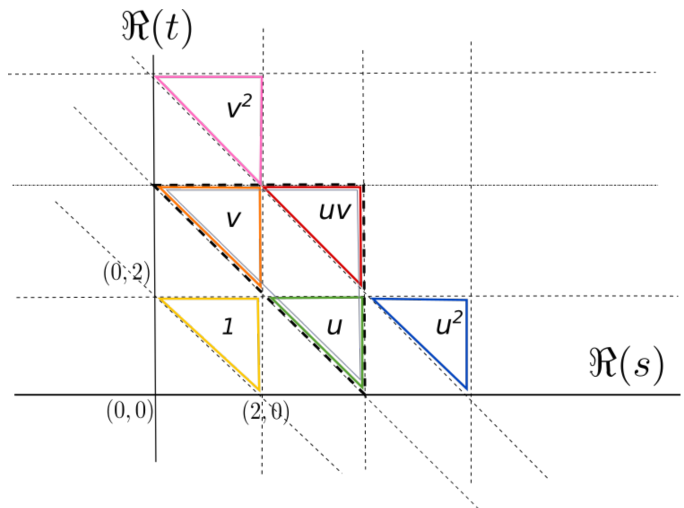

Figure C.2: The fundamental domains for the Mellin transform of $R\mathcal{H}$.

Having stated the results for the two sides of (C.4) (the “unmassaged” lhs, whose Mellin transform is given by (C.3), and the “massaged” rhs, where the Mellin transform of $\mathcal{H}$ is given by (C.8)), we will now try to match them. Compared to the supergravity answer, there are three more size-two triangles on the right side. They are in the colors of yellow, pink and blue, and are

[1]In addition to the previously defined red, green, orange triangles, there are also size-two pink $\{(0,6),(2,4),(2,6)\}$, yellow $\{(0,2),(2,0),(2,2)\}$ and blue $\{(4,2),(6,0),(6,2)\}$ triangles.

respectively due to the shifts caused by the terms τ, $V^2\sigma$ and $U^2\sigma\tau$. Using the regularization procedure we introduced in Section 4.1.3, they can be eliminated by combining with terms from the other triangles that we want to keep. Let us now describe in detail how this can be done.

We first pay attention to the terms multiplied by τ in R. We will combine it with terms multiplied by $-\tau V$ and $-\tau U$ from R. Naively the three shifted domains will not overlap. Under the regularization, these three domains grow a small overlap and allows us to add the integrands once the contours have all been moved there

$$\tau(1-V-U)\mathcal{H} = -\frac{\tau}{N^2}\int_{\mathcal{C}_{(2,2),\epsilon}} dsdtU^{s/2}V^{t/2-2} \times \left[\frac{st-4}{2}+\frac{s+t-3}{2}\epsilon+\frac{\epsilon^2}{4}\right]$$
$$\times\Gamma[2-\frac{s}{2}]\Gamma[1-\frac{s}{2}]\Gamma[2-\frac{t}{2}]\Gamma[1-\frac{t}{2}]\Gamma[\frac{s+t+\epsilon}{2}-2]\Gamma[\frac{s+t+\epsilon}{2}-1]\ . \tag{C.10}$$

Here $\mathcal{C}_{(2,2),\epsilon}$ denotes that we put the contour inside the size-ϵ triangle (not shown in the picture) at $(2,2)$ shared by these three triangles. We now analyze the terms in this integral.

The ϵ^1 term is the same integral as the one that we have encountered in the proof of the identity. It is evaluated to give

$$-\tau\frac{4}{N^2}UV^{-1}\ . \tag{C.11}$$

The ϵ^2 term is easily seen to be zero. For the ϵ^0 term, we rewrite it as

$$\frac{st-4}{2}=\frac{1}{2}(s-2)(t-2)+(s-2)+(t-2)\ . \tag{C.12}$$

The point of this rewriting is that these zeros of $(s-2)$ and $(t-2)$ will cancel the same poles in the Gamma functions, such that one is allowed to "open up the boundaries" to enter a bigger domain. For example, consider the above term $(s-2)$. Its contour was originally placed at the size-ϵ domain at $(2,2)$ but now it can be moved into size-two green triangle because $(s-2)$ cancels the simple pole at $s=2$ from $\Gamma[1-\frac{s}{2}]$. Similarly the domain of the $\frac{1}{2}(s-2)(t-2)$ term can be extended to the size-four grey triangle and the $(t-2)$ term extended to the size-two orange triangle with the same reason.

On the other hand, for the σV^2 triangle, we will combine it into $\sigma(-V+V^2-UV)\mathcal{H}$. The goal of splitting the $\mathcal{O}(\epsilon^0)$ term here is to open up the boundaries into the orange, red and grey triangle and from the ϵ^1 term one will get a monomial $-\sigma\frac{4}{N^2}U$. For the $\sigma\tau U^2$ triangle, one combines into $\sigma\tau(-U+U^2-UV)\mathcal{H}$. The ϵ term from the rewriting generates a monomial $-\sigma\tau\frac{4}{N^2}U^2V^{-1}$. Already, collecting these monomials, one get $-\mathcal{G}_{\text{free,conn}}$, canceling precisely the free field part in the split formula.

To carry out the rest of the check, it is simplest to check by gathering terms with the same R-symmetry monomial. In the $p=2$ case one has six R-symmetry monomials and one can divide them into two groups: first check 1, σ^2 and τ^2, then τ, σ, $\sigma\tau$. In fact, checking just one term in each class is enough, because both the supergravity result and the result written in a split form have crossing symmetry. These two classes of monomials form two orbits under the S_3 crossing symmetry group. One will need also to use the above trick of using zeros to open up boundaries (or the opposite, use poles to close). But the here one will find it is only necessary to shrink or expand between the size-four grey triangle and a size-two orange, red, green triangles. Because the manipulation is from a finite-size domain to another finite-size domain, the contour will always have room to escape and one will never get additional terms from the "domain-pinching" mechanism. We performed this explicit check and found a perfect match.

Appendix D

$AdS_7 \times S^4$ Four-Point Functions and the $\mathcal{W}_{n\to\infty}$ algebra

Using the position space method, we computed in Section 2.3.1 the four-point functions for $AdS_7 \times S^4$ with $k = 2$ and $k = 3$. The results can be massaged into a form that is manifestly consistent with the solution of the superconformal Ward identity (4.56).

For $k = 2$, the result can be written as

$$\mathcal{A}_2 = \frac{\sigma\tau U^4}{n^3 V^2} + \frac{\sigma U^2}{n^3} + \frac{\tau U^2}{n^3 V^2} + \Upsilon \circ \left(\frac{U^5}{2n^3 V} \bar{D}_{7333} \right) \tag{D.1}$$

where the differential operator Υ was defined in (4.57). Upon performing the chiral algebra twist, we get a holomorphic correlator,

$$\mathcal{A}_2(\chi, \chi'; 1/\chi', 1/\chi') = \frac{2\chi^2((\chi-1)\chi+1)}{n^3(\chi-1)^2} \ . \tag{D.2}$$

Similarly, for $k = 3$ the answer can be written as

$$\mathcal{A}_3 = \mathcal{F}_3 + \Upsilon \circ \mathcal{H}_3 \ . \tag{D.3}$$

Here $\mathcal{F}_{3,\mathrm{conn}}$ is a simple rational function of the cross ratios,

$$\mathcal{F}_{3,\mathrm{conn}} = \frac{9}{4n^3} \left(\frac{\sigma\tau^2 U^6}{V^4} + \frac{\sigma^2\tau U^6}{V^2} + \sigma^2 U^4 + \frac{\tau^2 U^4}{V^4} + \sigma U^2 + \frac{\tau U^2}{V^2} \right) \ , \tag{D.4}$$

while $\mathcal{H}_3$ is given in terms of $\bar{D}$-functions by the following expression,

$$\mathcal{H}_3 = \frac{U^5}{48n^3V}\bigg(U^2(9\bar{D}_{9533} + 7\bar{D}_{9544} + 2\bar{D}_{9555}) + \sigma(9\bar{D}_{3539} + 7\bar{D}_{4549} + 2\bar{D}_{5559}) + \tau V^2(9\bar{D}_{3593} + 7\bar{D}_{4594} + 2\bar{D}_{5595})\bigg) \,. \tag{D.5}$$

After we twist the R-symmetry variables, we extract another holomorphic correlator

$$\mathcal{A}_3(\chi,\chi';1/\chi',1/\chi') = \frac{9\chi^2\left(2\chi^6 - 6\chi^5 + 9\chi^4 - 8\chi^3 + 9\chi^2 - 6\chi + 2\right)}{4n^3(\chi-1)^4} \,. \tag{D.6}$$

The above holomorphic correlators are conjectured be the four-point functions of $\mathcal{W}_{n\to\infty}$ algebra. In the rest of the appendix, we perform an independent 2d calculation to check this proposal. We will use the "holomorphic bootstrap" method of [107] to compute four-point functions of the $\mathcal{W}_{n\to\infty}$ algebra.

We start by recalling that in a chiral algebra the four-point function of identical quasi-primary operators,

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)\mathcal{O}_h(z_3)\mathcal{O}_h(z_4)\rangle = (z_{12}z_{34})^{-2h}\mathcal{F}(\chi)\,, \tag{D.7}$$

satisfies the following crossing equation

$$\mathcal{F}(\chi) = \chi^{2h}\mathcal{F}(1/\chi) = \chi^{2h}(1-\chi)^{-2h}\mathcal{F}(1-\chi)\,. \tag{D.8}$$

The function $\mathcal{F}(\chi)$ can be written as a sum of the $SL(2,\mathbb{R})$ blocks

$$\mathcal{F}(\chi) = \sum_i C^2_{hhi}\chi^{h_i}{}_2F_1(h_i,h_i;2h_i;\chi)\,. \tag{D.9}$$

Here C_{hhi} is the three-point coupling in

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)\mathcal{O}_{h_i}(z_3)\rangle = \frac{C_{hhi}}{z_{12}^{2h-h_i}z_{13}^{h_i}z_{23}^{h_i}}\,, \tag{D.10}$$

and all two-point functions are normalized to unity. Combining the crossing equation with the conformal block decomposition, we find that the singularities of $\mathcal{F}$ at $\chi = 1$ and $\chi \to \infty$ are

$$\begin{aligned}
\mathcal{F}(\chi) &= (-1)^{4h} \sum_i C_{hhi}^2 (1-\chi)^{h_i - 2h} {}_2F_1(h_i, h_i; 2h_i; 1-\chi)\chi^{2h} \\
&\approx \sum_{n=1}^{2h} \beta_n (1-\chi)^{-n} + \dots \qquad \text{as } \chi \to 1 \,, \\
\mathcal{F}(\chi) &= (-1)^{4h} \sum_i C_{hhi}^2 \chi^{2h-h_i} {}_2F_1(h_i, h_i; 2h_i; 1/\chi) \\
&\approx \sum_{n=1}^{2h} \beta_n \chi^n + \dots \qquad \text{as } \chi \to \infty \,.
\end{aligned} \tag{D.11}$$

Here α_n and β_n are computable numbers given the three-point functions coefficients C_{hhi} with $h_i < 2h$.

The meromorphic $\mathcal{F}(\chi)$ is completely determined by these singularities and by its value at $\chi = 0$,

$$\mathcal{F}(\chi) = 1 + \sum_{n=1}^{2h} \alpha_n \chi^n + \sum_{n=1}^{2h} \beta_n [(1-\chi)^{-n} - 1] \,. \tag{D.12}$$

Alternatively, we can cast $\mathcal{F}(\chi)$ into the following more convenient parameterization [107],

$$\mathcal{F}(\chi) = \sum_{n=0}^{[2h/3]} c_n \frac{\chi^{2n}(1-\chi+\chi^2)^{2h-3n}}{(1-\chi)^{2h-2n}} \,, \tag{D.13}$$

which is manifestly crossing-symmetric and has the same singularity structure, where the finitely many constants c_n are determined from the conformal block decomposition at $\chi = 0$.

$k = 2$

We first look at the $k = 2$ case which corresponds to $2d$ stress tensor. The general form of $\mathcal{F}(\chi)$ for $h = k = 2$ reads

$$\mathcal{F}(\chi) = \frac{c_0\,(\chi^2 - \chi + 1)^4 + c_1 (1-\chi)^2 \chi^2\,(\chi^2 - \chi + 1)}{(1-\chi)^4} \,. \tag{D.14}$$

To fix the constants c_n we only need to match with the OPE coefficients of $TT1$ and TTT

$$C_{TT1} = 1 \,, \qquad C_{TTT} = 2^{3/2} c^{-1/2} \,. \tag{D.15}$$

We find

$$c_0 = 1 \,, \qquad c_1 = \frac{8}{c} - 4 \,. \tag{D.16}$$

In the $1/c$ expansion, the four-point function with the above solution of c_n simply reads

$$\mathcal{F}(\chi) = 1 + \chi^4 + \frac{\chi^4}{(1-\chi)^4} + \frac{1}{c}\left(\frac{8\chi^2(1-\chi+\chi^2)}{(1-\chi)^2}\right) + \mathcal{O}(c^{-2}) \,. \tag{D.17}$$

Notice the leading term is nothing but the disconnected piece of the full four-point function under the chiral algebra twist and is anticipated from the large-c factorization. The subleading term in $1/c$ on the other hand reproduces precisely the holomorphic four-point function we obtained from the supergravity computation, upon recalling that $c \sim 4n^3$ in the large n limit.

$k = 3$

In the case of $k = 3$, the general form of $\mathcal{F}(\chi)$ admits three parameters,

$$\mathcal{F}(\chi) = \frac{c_0\,(\chi^2-\chi+1)^6 + c_1(1-\chi)^2\chi^2\,(\chi^2-\chi+1)^3 + c_2(1-\chi)^4\chi^4}{(1-\chi)^6} \,. \tag{D.18}$$

This chiral four-point function is to be matched with the W_3W_31 coefficient as well as W_3W_3T, $W_3W_3W_4$ coefficients which are given in [91, 108]. The end result is

$$c_0 = 1 \,, \qquad c_1 = \frac{18}{c} - 6 \,, \qquad c_2 = \frac{3(c-9)}{c} \,. \tag{D.19}$$

The large c expansion of $\mathcal{F}(\chi)$ gives

$$\begin{aligned}\mathcal{F}(\chi) &= 1 + \chi^6 + \frac{\chi^6}{(1-\chi)^6} \\ &\quad + \frac{1}{c}\left(\frac{9\chi^2\,(2\chi^6 - 6\chi^5 + 9\chi^4 - 8\chi^3 + 9\chi^2 - 6\chi + 2)}{(\chi-1)^4}\right) + \mathcal{O}(c^{-2}) \,,\end{aligned} \tag{D.20}$$

matching again the supergravity result.

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